

CHAPTER 8

CONCLUSIONS

8.1 General Conclusions

In this book, we have attempted to give, at a fairly advanced level of rigor, a unified treatment of current methodologies for the design and analysis of adaptive control algorithms.

First, we presented several schemes for the adaptive identification and control of linear time invariant systems. An output error scheme, an input error scheme, and an indirect scheme were derived in a unified framework. While all the schemes were shown to be globally stable, the assumptions that went into the derivation of the schemes were quite different. For instance, the input error adaptive control scheme did not require a strictly positive real (SPR) condition for the reference model. This also had implications for the transient behavior of the adaptive systems.

A major goal of this book has been the presentation of a number of recent techniques for analyzing the stability, parameter convergence and robustness of the complicated nonlinear dynamics inherent in the adaptive algorithms. For the stability proofs, we presented a sequence of lemmas drawn from the literature on input-output L_p stability. For the parameter convergence proofs, we used results from generalized harmonic analysis, and extracted frequency-domain conditions. For the study of robustness, we exploited Lyapunov and averaging methods. We feel that a complete mastery of these techniques will lay the groundwork for future studies of adaptive systems.

While we did not deal explicitly with discrete time systems, our presentation of the continuous time results may be transcribed to the discrete time case with not much difficulty. The operator relationships that were used for continuous time systems (L_p spaces) also hold true for discrete time systems (l_p spaces). In fact, many derivations may be simplified in the discrete time case because continuity conditions (such as the regularity of signals) are then automatically satisfied.

Averaging techniques have proved extremely useful and it is likely that important developments will still follow from their use. It is interesting to note that the two-time scale approximation was not only fundamental to the application of averaging methods to convergence (Chapter 4) and to robustness (Chapter 5), but was also underlying in the proofs of exponential convergence (Chapter 2), and global stability (Chapter 3). This highlights the separation between adaptation and control, and makes the connections between direct and indirect adaptive control more obvious.

Methods for the analysis of adaptive systems were a focal point of this book. As was observed in Chapter 5, algorithms that are stable for some inputs may be unstable for others. While simulations are extremely valuable to illustrate a point, they are useless to prove any global behavior of the adaptive algorithm. This is a crucial consequence of the nonlinearity of the adaptive systems, that makes rigorous analysis techniques essential to progress in the area.

8.2 Future Research

Adaptive control is a very active area of research, and there is a great deal more to be done. The area of robustness is essential to successful applications, and since the work of Rohrs *et al*, it has been understood that the questions of robustness for adaptive systems are very different from the same questions for linear time-invariant systems. This is due in great part to the *dual control* aspect of adaptive systems: the reference input plays a role in determining the convergence and robustness by providing excitation to the identification loop. A major problem remains to quantify robustness for adaptive systems. Current theory does not allow for the comparison of the robustness of different adaptive systems, and the relation to non-adaptive robustness concepts. Closer connections will probably emerge from the application of averaging methods, and from the frequency-domain results that they lead to.

Besides these fundamental questions of analysis, much remains to be done to precisely define *design methodologies* for robust adaptive systems and in particular a better understanding of which algorithms are more robust. Indeed, although the adaptive systems discussed in this book have identical stability properties in the ideal case, there is

evidence that their behavior is drastically different in the presence of unmodeled dynamics. A better understanding of which algorithms are more robust will also help in deriving guidelines for the improved design of robust algorithms.

While we have extensively discussed the analysis of adaptive systems, we also feel that great strides in this area will come from experiences in implementing the algorithms on several classes of systems. With the advent of microprocessors, and of today's multi-processor environments, complicated algorithms can now be implemented at very high sample rates. The years to come will see a proliferation of techniques to effectively map these adaptive algorithms onto multiprocessor control architectures. There is a great deal of excitement in the control community at large over the emergence of such custom multiprocessor control architectures as CONDOR (Narasimhan *et al* [1988]) and NYMPH (Chen *et al* [1986]). In turn, such advances will make it possible to exploit adaptive techniques on high bandwidth systems such as flexible space structures, aircraft flight control systems, light weight robot manipulators, and the like. While past successes of adaptive control have been on systems of rather low bandwidth and benign dynamics, the future years are going to be ones of experimentation on more challenging systems.

Two other areas that promise explosive growth in the years to come are adaptive control of multi-input multi-output (MIMO) systems, and adaptive control of nonlinear systems, explicitly those linearizable by state feedback. We presented in this book what we feel is the tip of the iceberg in these areas. More needs to be understood about the sort of prior information needed for MIMO adaptive systems. Conversely, the incorporation of various forms of prior knowledge into black-box models of MIMO systems also needs to be studied. Adaptive control for MIMO systems is especially attractive because the traditional and heuristic techniques for SISO systems quickly fall apart when strong cross-couplings appear. Note also that research in the identification of MIMO systems is also relevant to nonadaptive algorithms, which are largely dependent on the knowledge of a process model, and of its uncertainty. One may hope that the recently introduced averaging techniques will help to better connect the frequency-domain properties of adaptive and nonadaptive systems.

A very large class of nonlinear systems is explicitly linearizable by state feedback. The chief difficulty with implementing the linearizing control law is the imprecise knowledge of the nonlinear functions in the dynamics, some of which are often specified in table look-up form. Adaptation then has a role in helping identify the nonlinear functions on-line to obtain asymptotically the correct linearizing control law. This

approach was discussed in this book, but it is still in its early development. However, we have found it valuable in the implementation of an adaptive controller for an industrial robot (the Adept-I) and are currently working on a flight control system for a vertical take-off and landing aircraft (the Harrier).

In addition to all these exciting new directions of research in adaptive control, most of which are logical extensions and outgrowths of the developments presented in the previous chapters, we now present a few other new vistas which are not as obvious extensions.

A "Universal" Theory of Adaptive Control

While all the adaptive control algorithms developed in this book required assumptions on the plant—in the single-input single-output case, the order of the plant, the relative degree of the plant, the sign of the high-frequency gain, and the minimum phase property of the plant—it is interesting to ask if these assumptions are a minimal set of assumptions. Indeed, that these assumptions can be relaxed was established by Morse [1985, 1987], Mudgett and Morse [1985], Nussbaum [1983], and Martensson [1985] among others. Chief under the assumptions that could be relaxed was the one on the sign of the high-frequency gain.

There is a simple instance of these results which is in some sense representative of the whole family: consider the problem of adaptively stabilizing a first order linear plant of relative degree 1 with unknown gain k_p ; i.e.,

$$\dot{y}_p = -a_p y_p + k_p u \quad (8.2.1)$$

with k_p different from zero but otherwise unknown, and a_p unknown. If the sign of k_p is known and assumed positive, the adaptive control law

$$u = d_0(t) y_p \quad (8.2.2)$$

and

$$\dot{d}_0 = -y_p^2 \quad (8.2.3)$$

can be shown to yield $y_p \rightarrow 0$ as $t \rightarrow \infty$. Nussbaum [1983] proposed that if the sign of k_p is *unknown*, the control law (8.2.2) can be replaced by

$$u = d_0^2(t) \cos(d_0(t)) y_p \quad (8.2.4)$$

with (8.2.3) as before. He then showed that $y_p \rightarrow 0$ as $t \rightarrow \infty$, with $d_0(t)$ remaining bounded. Heuristically, the feedback gain $d_0^2 \cos(d_0)$ of (8.2.4) alternates in sign ("searches for the correct sign") as d_0 is

decreased monotonically (by (8.2.3)) until it is large enough and of the "correct sign" to stabilize the equation (8.2.1).

While the transient behavior of the algorithm (8.2.3), (8.2.4) is poor, the scheme has stimulated a great deal of interest to derive adaptive control schemes requiring a minimal set of assumptions on the plant (*universal controllers*). A further objective is to develop a unified framework which would subsume all the algorithms presented thus far. Adaptive systems may be seen as the interconnection of a plant, a parameterized controller, and adaptation law or tuner (cf. Morse [1988]). The parameterized controller is assumed to control the process, and the tuner assumed to tune the controller. Tuning is said to have taken place when a suitable tuning error goes to zero. The goal of a *universal theory* is to give a minimal set of assumptions on the process, the parameterized controller, and the tuner to guarantee global stability and asymptotic performance of the closed loop system. Further, the assumptions are to contain as special cases the algorithms presented thus far. Such a theory would be extremely valuable from a conceptual and intellectual standpoint.

Rule-Based, Expert and Learning Control Systems

As the discussions in Chapter 5 indicated, there is a great deal of work needed to implement a given adaptive algorithm, involving the use of heuristics, prior knowledge, and expertise about the system being controlled (such as the amount of noise, the order of the plant, the number of unknown parameters, the bandwidth of the parameters' variation...). This may be coded as several logic steps or rules, around the adaptive control algorithm. The resulting composite algorithm is often referred to as a *rule-based* control law, with the adaptation scheme being one of the rules. The design and evaluation of such composite systems is still an open area of research for nonadaptive as well as adaptive systems, although adaptive control algorithms form an especially attractive area of application.

One can conceive of a more complex scenario, in which the plant to be controlled cannot be easily modeled, either as a linear or nonlinear system because of the complexity of the physical processes involved. A controller then has to be built by codifying systematically into rules the experience gained from operating the system (this is referred to as querying and representation of expert knowledge). The rules then serve as a model of the plant from which the controller is constructed as a rule-based system, i.e. a conjunction of several logic steps and control algorithms. Such a composite design process is called a *rule-based expert controller design*. The sophistication and performance of the controller is dependent on the amount of detail in the model furnished by the

expert knowledge. Adaptation and learning in this framework consists in refining the rule-based model on the experience gained during the course of operation of the system.

While this framework is extremely attractive from a practical point of view, it is fair to say that no more than a few case studies of expert control have been implemented, and state of the art in learning for rule-based models is rudimentary. In the context of adaptive control, a very interesting study is found in Astrom *et al* [1986]. Adapted from their work is Figure 8.1, illustrating the structure of an expert control system using an adaptive algorithm.

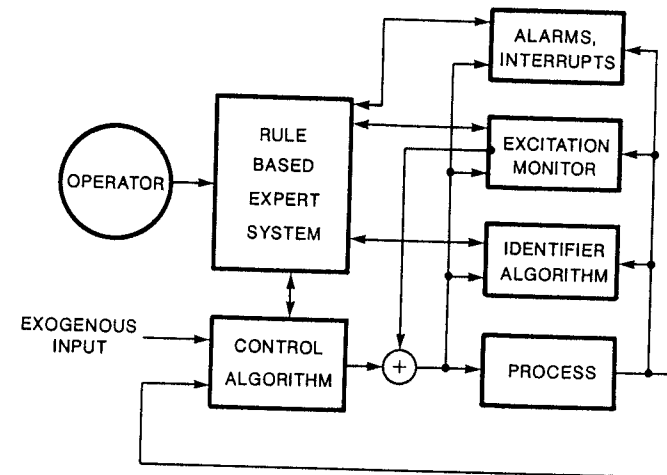


Figure 8.1: Expert Adaptive Control System

The rule-based system decides, based on the level of excitation, which of a library of identification algorithms to use and, if necessary, to inject new excitation. It also decides which of a family of control laws to use and communicates its inferencing procedures to the operator. A supervisor provides alarms and interrupts.

Adaptation, Learning, Connectionism and all those things...

While the topics in the title have the same general philosophical goals, namely, the understanding, modeling and control of a given process, the fields of identification and adaptive control have made the largest strides in becoming a design methodology by limiting their universe of discourse to a small (but practically meaningful) class of systems with linear or linearizable dynamics, and a finite dimensional state-space. Learning has, however, been merely parameter updating.

The goals of connectionism and neural networks (see for example Denker [1986] and the parallel distributed processing models) have been far more lofty: the universe of discourse includes human systems and the learning mimics our own human development. A few applications of this work have been made to problems of speech recognition, image recognition, associative memory storage elements and the like, and it is interesting to note that the 'learning' algorithms implemented in the successful algorithms are remarkably reminiscent of the gradient type and least-squares type of update laws studied in this book. We feel, consequently, that in the years to come, there will be a confluence of the theories and techniques for learning.

APPENDIX

Proof of Lemma 1.4.2

Let

$$r(t) = \int_0^t a(\tau)x(\tau) d\tau \quad (\text{A1.4.1})$$

so that, by assumption

$$\dot{r}(t) = a(t)x(t) \leq a(t)r(t) + a(t)u(t) \quad (\text{A1.4.2})$$

that is, for some positive $s(t)$

$$\dot{r}(t) - a(t)r(t) - a(t)u(t) + s(t) = 0 \quad (\text{A1.4.3})$$

Solving the differential equation with $r(0) = 0$

$$r(t) = \int_0^t e^{\int_0^\tau a(\sigma) d\sigma} (a(\tau)u(\tau) - s(\tau)) d\tau \quad (\text{A1.4.4})$$

Since $\exp(\cdot)$ and $s(\cdot)$ are positive functions

$$r(t) \leq \int_0^t e^{\int_0^\tau a(\sigma) d\sigma} a(\tau)u(\tau) d\tau \quad (\text{A1.4.5})$$

By assumption $x(t) \leq r(t) + u(t)$ so that (1.4.11) follows. Inequality (1.4.12) is obtained by integrating (1.4.11) by parts. \square

Proof of Lemma 2.5.2

We consider the system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) \\ y(t) &= C(t)x(t)\end{aligned}\quad (\text{A2.5.1})$$

and the system under output injection

$$\begin{aligned}\dot{w}(t) &= [A(t) + K(t)C(t)]w(t) \\ z(t) &= C(t)w(t)\end{aligned}\quad (\text{A2.5.2})$$

where $x, w \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$, $K \in \mathbb{R}^{n \times m}$, and $y, z \in \mathbb{R}^m$.

It is sufficient to derive equations the inequalities giving β_1' , β_2' , β_3' .

Derivation of β_1'

Consider the trajectories $x(\tau)$ and $w(\tau)$, corresponding to systems (A2.5.1) and (A2.5.2) respectively, with identical initial conditions $x(t_0) = w(t_0)$. Then

$$w(\tau) - x(\tau) = \int_{t_0}^{\tau} \Phi(\tau, \sigma) K(\sigma) C(\sigma) w(\sigma) d\sigma \quad (\text{A2.5.3})$$

Let $e(\sigma) = K(\sigma)C(\sigma)w(\sigma) / |K(\sigma)C(\sigma)w(\sigma)| \in \mathbb{R}^n$, so that

$$\begin{aligned}|C(\tau)(w(\tau) - x(\tau))|^2 &= \left| \int_{t_0}^{\tau} C(\tau) \Phi(\tau, \sigma) K(\sigma) C(\sigma) w(\sigma) d\sigma \right|^2 \\ &\leq \left[\int_{t_0}^{\tau} |C(\tau) \Phi(\tau, \sigma) e(\sigma)| \|K(\sigma)\| |C(\sigma) w(\sigma)| d\sigma \right]^2 \\ &\leq \int_{t_0}^{\tau} |C(\nu) w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau) \Phi(\tau, \sigma) e(\sigma)|^2 \|k(\sigma)\|^2 d\sigma\end{aligned}\quad (\text{A2.5.4})$$

using the definition of the induced norm and Schwartz inequality. On the other hand, using the triangular inequality

$$\left[\int_{t_0}^{t_0 + \delta} |C(\tau) w(\tau)|^2 d\tau \right]^{\frac{1}{2}} \geq \left[\int_{t_0}^{t_0 + \delta} |C(\tau) x(\tau)|^2 d\tau \right]^{\frac{1}{2}}$$

$$- \left[\int_{t_0}^{t_0 + \delta} |C(\tau)(w(\tau) - x(\tau))|^2 d\tau \right]^{\frac{1}{2}} \quad (\text{A2.5.5})$$

so that, using (A2.5.4), and the UCO of the original system

$$\begin{aligned}\left[\int_{t_0}^{t_0 + \delta} |C(\tau) w(\tau)|^2 d\tau \right]^{\frac{1}{2}} &\geq \sqrt{\beta_1} |w(t_0)| \\ - \left[\int_{t_0}^{t_0 + \delta} \int_{t_0}^{\tau} |C(\nu) w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau) \Phi(\tau, \sigma) e(\sigma)|^2 \|K(\sigma)\|^2 d\sigma d\tau \right]^{\frac{1}{2}} \\ &\geq \sqrt{\beta_1} |w(t_0)| - \left[\int_{t_0}^{t_0 + \delta} |C(\nu) w(\nu)|^2 d\nu \right]^{\frac{1}{2}} \\ &\cdot \left[\int_{t_0}^{t_0 + \delta} \int_{t_0}^{\tau} \|K(\sigma)\|^2 |C(\tau) \Phi(\tau, \sigma) e(\sigma)|^2 d\sigma d\tau \right]^{\frac{1}{2}}\end{aligned}\quad (\text{A2.5.6})$$

Changing the order of integration, the integral in the last parenthesis becomes

$$\int_{t_0}^{t_0 + \delta} \|K(\sigma)\|^2 \int_{\sigma}^{t_0 + \delta} |C(\tau) \Phi(\tau, \sigma) e(\sigma)|^2 d\tau d\sigma \quad (\text{A2.5.7})$$

Note that $t_0 + \delta - \sigma \leq \delta$, $|e(\sigma)| = 1$, while $\Phi(\tau, \sigma)e(\sigma)$ is the solution of system (A2.5.1) starting at $e(\sigma)$. Therefore, using the UCO property on the original system, and the condition on $K(\cdot)$, (A2.5.7) becomes

$$\int_{t_0}^{t_0 + \delta} \|K(\sigma)\|^2 \int_{\sigma}^{t_0 + \delta} |C(\tau) \Phi(\tau, \sigma) e(\sigma)|^2 d\tau d\sigma \leq k_\delta \beta_2 \quad (\text{A2.5.8})$$

Inequality (2.5.7) follows directly from (A2.5.6) and (A2.5.8).

Derivation of β_2'

We use a similar procedure, using (A2.5.4)

$$|C(\tau) w(\tau)|^2 \leq |C(\tau) x(\tau)|^2$$

$$\begin{aligned}
& + \left| \int_{t_0}^{\tau} C(\tau) \Phi(\tau, \sigma) K(\sigma) C(\sigma) w(\sigma) d\sigma \right|^2 \\
& \leq |C(\tau)x(\tau)|^2 \\
& + \left[\int_{t_0}^{\tau} |C(\sigma)w(\sigma)| |C(\tau)\Phi(\tau, \sigma)e(\sigma)| \|K(\sigma)\| d\sigma \right]^2 \\
& \leq |C(\tau)x(\tau)|^2 \\
& + \int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \int_{t_0}^{\tau} |C(\tau)\Phi(\tau, \sigma)e(\sigma)|^2 \|K(\sigma)\|^2 d\sigma \quad (\text{A2.5.9})
\end{aligned}$$

and, for all $t_0 \leq t \leq t_0 + \delta$

$$\begin{aligned}
\int_{t_0}^t |C(\tau)w(\tau)|^2 d\tau & \leq \int_{t_0}^{t_0+\delta} |C(\tau)x(\tau)|^2 d\tau + \int_{t_0}^t \int_{t_0}^{\tau} |C(\nu)w(\nu)|^2 d\nu \\
& \cdot \int_{t_0}^{\tau} |C(\tau)\Phi(\tau, \sigma)e(\sigma)|^2 \|K(\sigma)\|^2 d\sigma d\tau \quad (\text{A2.5.10})
\end{aligned}$$

and, using the Bellman-Gronwall lemma (lemma 1.4.2), together with the UCO of the original system

$$\begin{aligned}
\int_{t_0}^t |C(\tau)w(\tau)|^2 d\tau & \leq \beta_2 |w(t_0)|^2 \\
& \exp \left[\int_{t_0}^t \int_{t_0}^{\tau} |C(\tau)\Phi(\tau, \sigma)e(\sigma)|^2 \|K(\sigma)\|^2 d\sigma d\tau \right] \quad (\text{A2.5.11})
\end{aligned}$$

for all t , and in particular for $t = t_0 + \delta$.

The integral in the exponential can be transformed, by changing the order of integration, as in (A2.5.8). Inequality (2.5.8) follows directly from (A2.5.8) and (A2.5.11). \square

Proof of Lemma 2.6.6

We wish to prove that for some $\delta, \alpha_1, \alpha_2 > 0$, and for all x with $|x| = 1$

$$\alpha_2 \geq \int_{t_0}^{t_0+\delta} \left[(w^T + e^T)x \right]^2 d\tau \geq \alpha_1 \quad \text{for all } t_0 \geq 0 \quad (\text{A2.6.1})$$

By assumption, $e \in L_2$, so that $\int_0^{\infty} (e^T x)^2 d\tau \leq m$ for some $m \geq 0$. Since w is PE, there exist $\sigma, \beta_1, \beta_2 > 0$ such that

$$\beta_2 \geq \int_{t_0}^{t_0+\sigma} (w^T x)^2 d\tau \geq \beta_1 \quad \text{for all } t_0 \geq 0 \quad (\text{A2.6.2})$$

Let $\delta \geq \sigma \left[1 + \frac{m}{\beta_1} \right]$, $\alpha_1 = \beta_1$, $\alpha_2 = m + \beta_2 \left[1 + \frac{m}{\beta_1} \right]$ so that

$$\begin{aligned}
\int_{t_0}^{t_0+\delta} \left[(w^T + e^T)x \right]^2 d\tau & \geq \int_{t_0}^{t_0+\delta} (w^T x)^2 d\tau - \int_{t_0}^{t_0+\delta} (e^T x)^2 d\tau \\
& \geq \beta_1 \left[1 + \frac{m}{\beta_1} \right] - m = \alpha_1 \quad (\text{A2.6.3})
\end{aligned}$$

and

$$\begin{aligned}
\int_{t_0}^{t_0+\delta} \left[(w^T + e^T)x \right]^2 d\tau & \leq \int_{t_0}^{t_0+\delta} (w^T x)^2 d\tau + \int_{t_0}^{t_0+\delta} (e^T x)^2 d\tau \\
& \leq \beta_2 \left[1 + \frac{m}{\beta_1} \right] + m = \alpha_2 \quad (\text{A2.6.4})
\end{aligned}$$

\square

Proof of Lemma 2.6.7

We wish to prove that for some $\delta, \alpha_1, \alpha_2 > 0$, and for all x with $|x| = 1$

$$\alpha_2 \geq \int_{t_0}^{t_0+\delta} \left[\hat{H}(w^T)x \right]^2 d\tau \geq \alpha_1 \quad \text{for all } t_0 \geq 0 \quad (\text{A2.6.5})$$

Denote $u = w^T x$ and $y = \hat{H}(u) = \hat{H}(w^T x) = \hat{H}(w^T)x$ (where the last inequality is true because x does not depend on t). We thus wish to show that

$$\alpha_2 \geq \int_{t_0}^{t_0+\delta} y^2(\tau) d\tau \geq \alpha_1 \quad \text{for all } t_0 \geq 0 \quad (\text{A2.6.6})$$

Since w is PE, there exists $\sigma, \beta_1, \beta_2 > 0$ such that

$$\beta_2 \geq \int_{t_0}^{t_0+\sigma} u^2(\tau) d\tau \geq \beta_1 \quad \text{for all } t_0 \geq 0 \quad (\text{A2.6.7})$$

In this form, the problem appears on the relationship between truncated L_2 norms of the input and output of a stable, minimum phase LTI system. Similar problems are addressed in Section 3.6, and we will therefore use results from lemmas in that section.

Let $\delta = m\sigma$, where m is an integer to be defined later. Since u is bounded, and $y = \hat{H}(u)$, it follows that y is bounded (lemma 3.6.1) and the upper bound in (A2.6.6) is satisfied. The lower bound is obtained now, by inverting \hat{H} in a similar way as is used in the proof of lemma 3.6.2. We let

$$\hat{z}(s) = \frac{a^r}{(s+a)^r} \hat{u}(s) \quad (\text{A2.6.8})$$

where $a > 0$ will be defined later, and r is the relative degree of $\hat{H}(s)$. Thus

$$\hat{y}(s) = \frac{(s+a)^r}{a^r} \hat{H}(s) \hat{z}(s) \quad (\text{A2.6.9})$$

The transfer function from $\hat{z}(s)$ to $\hat{y}(s)$ has relative degree 0. Being minimum phase, it has a proper and stable inverse. By lemma 3.6.1, there exist $k_1, k_2 \geq 0$ such that

$$\int_{t_0}^{t_0+\delta} z^2(\tau) d\tau \leq k_1 \int_{t_0}^{t_0+\delta} y^2(\tau) d\tau + k_2 \quad (\text{A2.6.10})$$

Since \dot{u} is bounded

$$\int_{t_0}^{t_0+\delta} \dot{u}^2(\tau) d\tau \leq k_3 \delta \quad (\text{A2.6.11})$$

for some $k_3 \geq 0$. Using the results in the proof of lemma 3.6.2 ((A3.6.14)), we can also show that, with the properties of the transfer function $a^r / (s+a)^r$

$$\int_{t_0}^{t_0+\delta} u^2(\tau) d\tau \leq \int_{t_0}^{t_0+\delta} z^2(\tau) d\tau + \frac{r}{a} k_3 \delta + k_4 \quad (\text{A2.6.12})$$

where k_4 is another constant due to initial conditions. It follows that

$$\begin{aligned} \int_{t_0}^{t_0+\delta} y^2(\tau) d\tau &\geq \frac{1}{k_1} \left[\int_{t_0}^{t_0+\delta} u^2(\tau) d\tau - \frac{r}{a} k_3 \delta - k_2 - k_4 \right] \\ &\geq \frac{1}{k_1} \left[m \left(\beta_1 - \frac{r}{a} k_3 \sigma \right) - k_2 - k_4 \right] \end{aligned} \quad (\text{A2.6.13})$$

Note that r/a is arbitrary, and although k_1 depends on r/a , the constants β_1, k_3 , and σ do not. Consequently, we can let r/a sufficiently small that $\beta_1 - (r/a)k_3\sigma \geq \beta_1/2$. We can also let m be sufficiently large that $m\beta_1/2 - k_2 - k_4 \geq \beta_1$. Then the lower bound in (A2.6.6) is satisfied with

$$\alpha_1 = \frac{\beta_1}{k_1} \quad (\text{A2.6.14})$$

□

Proof of Lemma 3.6.2

The proof of lemma 3.6.2 relies on the auxiliary lemma presented hereafter.

Auxiliary Lemma

Consider the transfer function

$$\hat{K}(s) = \frac{a^r}{(s+a)^r} \quad a > 0 \quad (\text{A3.6.1})$$

where r is a positive integer.

Let $k(t)$ be the corresponding impulse response and define

$$g(t-\tau) = \int_{t-\tau}^{\infty} k(\sigma) d\sigma = \int_{-\infty}^{\tau} k(t-\sigma) d\sigma \quad t-\tau \geq 0 \quad (\text{A3.6.2})$$

Then

$$k(t) = \frac{a^r}{(r-1)!} t^{r-1} e^{-at} \quad t \geq 0 \quad (\text{A3.6.3})$$

and $k(t) = 0$ for $t < 0$. It follows that $k(t) \geq 0$ for all t , and

$$\|k\|_1 = \int_0^{\infty} k(\sigma) d\sigma = \int_{-\infty}^t k(t-\sigma) d\sigma = 1 \quad (\text{A3.6.4})$$

Similarly

$r = k$

$$g(t) = e^{-at} \sum_{k=1}^r \frac{(t^r - k)^r}{(t-k)!} a^{r-k} \quad t \geq 0 \quad (\text{A3.6.5})$$

and $g(t) = 0$ for $t < 0$. It follows that $g(t) \geq 0$ for all t , and

$$\|g\|_1 = \int_0^\infty g(\sigma) d\sigma = \int_{-\infty}^t g(t-\sigma) d\sigma = \frac{r}{a} \quad (\text{A3.6.6})$$

□

We are now ready to prove lemma 3.6.2. Let r be the relative degree of \hat{H} , and

$$\hat{z}(s) = \frac{a^r}{(s+a)^r} \hat{u}(s) \quad (\text{A3.6.7})$$

where $a > 0$ is an arbitrary constant to be defined later. Using (A3.6.7)

$$\hat{y}(s) = \frac{(s+a)^r}{a^r} \hat{H}(s) \hat{z}(s) \quad (\text{A3.6.8})$$

Since the transfer function from $\hat{z}(s)$ to $\hat{y}(s)$ has relative degree 0 and is minimum phase, it has a proper and stable inverse. By lemma 3.6.1

$$\|z_t\|_p \leq b_1 \|y_t\|_p + b_2 \quad (\text{A3.6.9})$$

We will prove that

$$\|u_t\|_p \leq c_1 \|z_t\|_p + c_2 \quad (\text{A3.6.10})$$

so that the lemma will be verified with $a_1 = c_1 b_1$, $a_2 = c_1 b_2 + c_2$.

Derivation of (A3.6.10)

We have that

$$z(t) = \epsilon(t) + \int_0^t k(t-\tau) u(\tau) d\tau \quad (\text{A3.6.11})$$

where $\epsilon(t)$ is an exponentially decaying term due to the initial conditions, and $k(t)$ is the impulse response corresponding to the transfer function in (A3.6.7) (derived in the auxiliary lemma). Integrate (A3.6.11) by parts to obtain

$$z(t) = \epsilon(t) + u(t) \int_{-\infty}^t k(t-\sigma) d\sigma - u(0) \int_{-\infty}^0 k(t-\sigma) d\sigma$$

When $r=0$, $u(t) = z(t)$ so that (A3.6.10) is trivially true. For $r > 0$, ...

$$- \int_0^t \left[\int_{-\infty}^{\tau} k(t-\sigma) d\sigma \right] \dot{u}(\tau) d\tau \quad (\text{A3.6.12})$$

Using the results of the auxiliary lemma

$$z(t) = \epsilon(t) + u(t) - u(0)g(t) - \int_0^t g(t-\tau) \dot{u}(\tau) d\tau \quad (\text{A3.6.13})$$

Since $g(t)$ is exponentially decaying, $u(0)g(t)$ can be included in $\epsilon(t)$. Also, using again the auxiliary lemma, together with lemma 3.6.1, and then the assumption on \dot{u} , it follows that

$$\begin{aligned} \|u_t\|_p &\leq \|z_t\|_p + \|\epsilon_t\|_p + \frac{r}{a} \|\dot{u}_t\|_p \\ &\leq \|z_t\|_p + \|\epsilon_t\|_p + \frac{r}{a} k_1 \|u_t\|_p + \frac{r}{a} k_2 \end{aligned} \quad (\text{A3.6.14})$$

Since a is arbitrary, let it be sufficiently large that $\frac{r}{a} k_1 < 1$. Consequently,

$$\begin{aligned} \|u_t\|_p &\leq \frac{1}{1 - \frac{r}{a} k_1} \|z_t\|_p + \frac{\|\epsilon\|_p + \frac{r}{a} k_2}{1 - \frac{r}{a} k_1} \\ &:= c_1 \|z_t\|_p + c_2 \end{aligned} \quad (\text{A3.6.15})$$

□

Proof of Corollary 3.6.3

(a) From lemma 3.6.2.

(b) Since \hat{H} is strictly proper, both y and \dot{y} are bounded.

(c) We have that $y = \hat{H}(u)$ and $\dot{y} = \hat{H}(\dot{u})$. Using successively lemma 3.6.1, the regularity of u , and lemma 3.6.2, it follows that for some constants k_1, \dots, k_6

$$\begin{aligned} |\dot{y}| &\leq k_1 \|\dot{u}_t\|_\infty + k_2 \\ &\leq k_3 \|u_t\|_\infty + k_4 \\ &\leq k_5 \|y_t\|_\infty + k_6 \end{aligned} \quad (\text{A3.6.16})$$

The proof can easily be extended to the vector case. □

Proof of Lemma 3.6.4

Let

$$\hat{H}(s) = h_0 + \hat{H}_1(s) \quad (\text{A3.6.17})$$

where \hat{H}_1 is strictly proper (and stable). Let h_1 be the impulse response corresponding to \hat{H}_1 . The output $y(t)$ is given by

$$y(t) = \epsilon(t) + h_0 u(t) + \int_0^t h_1(t-\tau) u(\tau) d\tau \quad (\text{A3.6.18})$$

where $\epsilon(t)$ is due to the initial conditions. Inequality (3.6.9) follows, if we define

$$\gamma_1(t) := |h_0| \beta_1(t) + \int_0^t |h_1(t-\tau)| \beta_1(\tau) d\tau \quad (\text{A3.6.19})$$

and

$$\gamma_2(t) := |\epsilon(t)| + |h_0| \beta_2(t) + \int_0^t |h_1(t-\tau)| \beta_2(\tau) d\tau \quad (\text{A3.6.20})$$

Since $\epsilon \in L_2$ and $h_1 \in L_1 \cap L_\infty$, we also have that $|\epsilon| \in L_2$, $|h_1| \in L_1 \cap L_\infty$. Since $\beta_1, \beta_2 \in L_2$, it follows that the last term of (A3.6.19) and similarly the last term of (A3.6.20) belong to $L_2 \cap L_\infty$, and go to zero as $t \rightarrow \infty$ (see e.g., Desoer & Vidyasagar [1975], exercise 5, p. 242). The conclusions follow directly from this observation. \square

Proof of Lemma 3.6.5

Let $[A, b, c^T, d]$ be a minimal realization of \hat{H} , with $A \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^m$, and $d \in \mathbb{R}$. Let $x: \mathbb{R}_+ \rightarrow \mathbb{R}^m$, and $y_1: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \dot{x} &= Ax + b(w^T \phi) \\ y_1 &= c^T x \end{aligned} \quad (\text{A3.6.21})$$

and $W: \mathbb{R}_+ \rightarrow \mathbb{R}^{m \times n}$, $y_2: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \dot{W} &= AW + bw^T \\ y_2 &= c^T W \phi \end{aligned} \quad (\text{A3.6.22})$$

Thus

$$\hat{H}(w^T \phi) = y_1 + d(w^T \phi) \quad \hat{H}(w^T \phi) = y_2 + (d w^T) \phi \quad (\text{A3.6.23})$$

Since

$$\frac{d}{dt}(W\phi) = \dot{W}\phi + W\dot{\phi} = AW\phi + bw^T\phi + W\dot{\phi} \quad (\text{A3.6.24})$$

it follows that

$$\begin{aligned} \frac{d}{dt}(x - W\phi) &= A(x - W\phi) - W\dot{\phi} \\ y_1 - y_2 &= c^T(x - W\phi) \end{aligned} \quad (\text{A3.6.25})$$

The result then follows since

$$\hat{H}(w^T \phi) - \hat{H}(w^T) \phi = y_1 - y_2 = \hat{H}_c(W\dot{\phi}) = \hat{H}_c(\hat{H}_b(w^T) \dot{\phi}) \quad (\text{A3.6.26})$$

 \square **Proof of Theorem 3.7.3**

The proof follows the steps of the proof of theorem 3.7.1 and is only sketched here.

(a) *Derive properties of the identifier that are independent of the boundedness of the regressor*

The properties of the identifier are the standard properties obtained in theorems 2.4.1–2.4.4

$$\begin{aligned} |\psi^T(t) \tilde{w}(t)| &= |\beta(t)| \|\tilde{w}_t\|_\infty + |\beta(t)| \\ \beta &\in L_2 \cap L_\infty \\ \psi &\in L_\infty \quad \dot{\psi} \in L_2 \cap L_\infty \\ a_{m+1}(t) &\geq k_{\min} > 0 \quad \text{for all } t \geq 0 \end{aligned} \quad (\text{A3.7.1})$$

The inequality for $a_{m+1}(t)$ follows from the use of the projection in the update law.

We also noted, in Section 3.3, that if π is bounded and a_{m+1} is bounded away from zero, then θ is also bounded, and the transformation has bounded derivatives. The vector q of coefficients of the polynomial \hat{q} is also bounded. By definition of the transformation, $\theta(\pi^*) = \theta^*$. Therefore, $\psi \in L_\infty$, $\dot{\psi} \in L_2 \cap L_\infty$ implies that $\phi \in L_\infty$, $\dot{\phi} \in L_2 \cap L_\infty$. Also, we have that $(k_m / \|a_{m+1}\|_\infty) \leq c_0(t) \leq k_m / k_{\min}$, for all $t \geq 0$.

(b) *Express the system states and inputs in term of the control error*
As in theorem 3.7.1.

(c) *Relate the identifier error to the control error*

We first establish an equality of ratios of polynomials, then we transform it to an operator equality. Using a similar approach as in the comments before the proof, we have that

$$\begin{aligned}\hat{q}\hat{a} - \hat{q}\hat{a}^* &= a_{m+1}(\hat{\lambda} - \hat{c}) - \hat{q}k_p\hat{n}_p \\ &= -a_{m+1}(\hat{c} - \hat{c}^*) + a_{m+1}(\hat{\lambda} - \hat{c}^*) - k_p\hat{q}\hat{n}_p \\ &= -a_{m+1}(\hat{c} - \hat{c}^*) + (a_{m+1}\hat{q}^* - k_p\hat{q})\hat{n}_p\end{aligned}\quad (\text{A3.7.2})$$

$$\begin{aligned}\hat{q}\hat{b} - \hat{q}\hat{b}^* &= \hat{q}\hat{\lambda} - a_{m+1}\hat{d} - \hat{\lambda}_0\hat{d}_m - \hat{q}\hat{\lambda} + \hat{q}\hat{d}_p \\ &= -a_{m+1}(\hat{d} - \hat{d}^*) + (-a_{m+1}\hat{d}^* - \hat{\lambda}_0\hat{d}_m + \hat{q}\hat{d}_p) \\ &= -a_{m+1}(\hat{d} - \hat{d}^*) \\ &\quad + \left[-\frac{a_{m+1}}{k_p}\hat{q}^* + \frac{a_{m+1}}{k_p}\hat{\lambda}_0\frac{\hat{d}_m}{\hat{d}_p} + \hat{q} - \hat{\lambda}_0\frac{\hat{d}_m}{\hat{d}_p} \right] \hat{d}_p\end{aligned}\quad (\text{A3.7.3})$$

Therefore

$$\begin{aligned}\frac{\hat{q}}{\hat{\lambda}_0} \left[\frac{\hat{a} - \hat{a}^*}{\hat{\lambda}} \right] + \frac{\hat{q}}{\hat{\lambda}_0} \left[\frac{\hat{b} - \hat{b}^*}{\hat{\lambda}} \right] \frac{k_p\hat{n}_p}{\hat{d}_p} \\ = -a_{m+1} \left[\frac{\hat{c} - \hat{c}^*}{\hat{\lambda}\hat{\lambda}_0} + \frac{\hat{d} - \hat{d}^*}{\hat{\lambda}\hat{\lambda}_0} \hat{P} \right] - (k_p - a_{m+1}) \frac{\hat{\lambda}_0}{\hat{\lambda}} \frac{\hat{d}_m\hat{n}_p}{\hat{d}_p} \frac{1}{\hat{\lambda}_0} \\ = -\frac{k_m}{c_0} \left[\frac{\hat{c} - \hat{c}^*}{\hat{\lambda}\hat{\lambda}_0} + \frac{\hat{d} - \hat{d}^*}{\hat{\lambda}\hat{\lambda}_0} \hat{P} + (c_0 - c_0^*)\hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} \right]\end{aligned}\quad (\text{A3.7.4})$$

where we divided by $\hat{\lambda}\hat{\lambda}_0$ to obtain proper stable transfer functions. The polynomial $\hat{\lambda}_0$ is Hurwitz and q is bounded, so that the operator $q^T \hat{s}_r / \hat{\lambda}_0$ is a bounded operator.

We now transform this polynomial equality into an operator equality as in the comments before the proof. Applying both sides of (A3.7.4) to u

$$-q^T \frac{\hat{s}_r}{\hat{\lambda}_0} (\tilde{w}^T) \psi = \frac{k_m}{c_0} \left[(c_0 - c_0^*) \hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} (u) + \bar{\phi}^T \frac{1}{\hat{\lambda}_0} (\bar{w}) \right]\quad (\text{A3.7.5})$$

The right-hand side is very reminiscent of the signal z obtained in the input error scheme. A filtered version of the signal $\hat{M}^{-1} \hat{P}(u) = r_p$ appears, instead of r , with the error $c_0 - c_0^*$. From proposition 3.3.1,

with $\hat{L} = \hat{\lambda}_0$ (cf. (3.3.17))

$$c_0^* \hat{M}^{-1} \hat{P} \frac{1}{\hat{\lambda}_0} (u) = \frac{1}{\hat{\lambda}_0} (u) - \frac{1}{\hat{\lambda}_0} (\bar{\theta}^{*T} \bar{w})\quad (\text{A3.7.6})$$

and since $u = c_0 r + \bar{\theta}^T \bar{w}$, it follows that

$$\hat{M}^{-1} \frac{\hat{P}}{\hat{\lambda}_0} (u) = \frac{1}{c_0^*} \left[\frac{1}{\hat{\lambda}_0} (c_0 r) + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) \right]\quad (\text{A3.7.7})$$

The right-hand side of (A3.7.5) becomes, using (A3.7.7) followed by the swapping lemma (and using the notation of the swapping lemma)

$$\begin{aligned}\frac{k_m}{c_0^*} \left[\frac{c_0 - c_0^*}{c_0} \frac{1}{\hat{\lambda}_0} (c_0 r) + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) + \frac{c_0^*}{c_0} \left[\bar{\phi}^T \frac{1}{\hat{\lambda}_0} (\bar{w}) - \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) \right] \right] \\ = \frac{k_m}{c_0^*} \left[\frac{1}{\hat{\lambda}_0} ((c_0 - c_0^*) r) - \hat{\Lambda}_{oc} \left[\hat{\Lambda}_{oc} (c_0 r) \left[\frac{c_0 - c_0^*}{c_0} \right] \right] \right. \\ \left. + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) - \frac{c_0^*}{c_0} \hat{\Lambda}_{oc} \left[\hat{\Lambda}_{ob} (\bar{w}^T) \bar{\phi} \right] \right]\end{aligned}\quad (\text{A3.7.8})$$

On the other hand, using again the swapping lemma, the left-hand side of (A3.7.5) becomes

$$q^T \frac{\hat{s}_r}{\hat{\lambda}_0} (\tilde{w}^T) \psi = q^T \frac{\hat{s}_r}{\hat{\lambda}_0} (\tilde{w}^T \psi) - q^T \hat{S}_{rc} (\hat{S}_{rb} (\tilde{w}^T) \psi)\quad (\text{A3.7.9})$$

where the transfer functions $\hat{\Lambda}_{ob}$, $\hat{\Lambda}_{oc}$, \hat{S}_{rb} , and \hat{S}_{rc} result from the application of the swapping lemma. The output error is then equal to (using (3.7.2), (A3.7.5), (A3.7.8), (A3.7.9))

$$\begin{aligned}y_p - y_m &= \frac{1}{c_0^*} \hat{M} \left[(c_0 - c_0^*) r + \bar{\phi}^T \bar{w} \right] \\ &= \frac{1}{k_m} \hat{M} \hat{\lambda}_0 \left[\frac{k_m}{c_0^*} \frac{1}{\hat{\lambda}_0} ((c_0 - c_0^*) r) + \frac{1}{\hat{\lambda}_0} (\bar{\phi}^T \bar{w}) \right]\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{k_m} \hat{M} \hat{\lambda}_0 \left[-q^T \frac{\hat{S}_r}{\hat{\lambda}_0} (\tilde{w}^T \psi) + q^T \hat{S}_{rc} \left[\hat{S}_{rb} (\tilde{w}^T \psi) \right] \right. \\
&\quad \left. + \frac{k_m}{c_0^*} \hat{\Lambda}_{oc} \left[\hat{\Lambda}_{ob}(c_0 r) \left[\frac{c_0 - c_0^*}{c_0} \right] \right] \right] \\
&\quad + \frac{k_m}{c_0} \hat{\Lambda}_{oc} \left[\hat{\Lambda}_{ob}(\bar{w}^T) \bar{\phi} \right] \quad (A3.7.10)
\end{aligned}$$

(d) *Establish the regularity of the signals*

As in theorem 3.7.1.

(e) *Stability Proof*

$\hat{M} \hat{\lambda}_0$ is a stable transfer function and since q^T is bounded, $q^T \hat{S}_r / \hat{\lambda}_0$ is a bounded operator. We showed that $\dot{\psi}$, $\dot{\phi}$, $\dot{c}_0 \in L_2$, so that, from (A3.7.10) and part (a), an inequality such as (3.7.19) can be obtained. As before \tilde{w} regular implies that $\beta \rightarrow 0$ as $t \rightarrow \infty$. The boundedness of all signals in the adaptive system then follows as in theorem 3.7.1. Similarly, $y_p - y_m \in L_2$ and tends to zero as $t \rightarrow \infty$. Since the relative degree of the transfer function from $u \rightarrow \tilde{w}$ is the same as the relative degree of \hat{P} , \hat{M} , and therefore \hat{L}^{-1} , the same result is true for $\tilde{w} - \tilde{w}_m$.

Proof of Lemma 4.2.1

Define

$$w_\epsilon(t, x) = \int_0^t d(\tau, x) e^{-\epsilon(t-\tau)} d\tau \quad (A4.2.1)$$

and

$$w_0(t, x) = \int_0^t d(\tau, x) d\tau \quad (A4.2.2)$$

From the assumptions

$$|w_0(t+t_0, x) - w_0(t_0, x)| \leq \gamma(t) \cdot t \quad (A4.2.3)$$

for all $t, t_0 \geq 0, x \in B_h$. Integrating (A4.2.1) by parts

$$w_\epsilon(t, x) = w_0(t, x) - \epsilon \int_0^t e^{-\epsilon(t-\tau)} w_0(\tau, x) d\tau \quad (A4.2.4)$$

Using the fact that

$$\epsilon \int_0^t e^{-\epsilon(t-\tau)} w_0(\tau, x) d\tau = w_0(t, x) - w_0(t, x) e^{-\epsilon t} \quad (A4.2.5)$$

(A4.2.4) can be rewritten as

$$w_\epsilon(t, x) = w_0(t, x) e^{-\epsilon t} + \epsilon \int_0^t e^{-\epsilon(t-\tau)} (w_0(t, x) - w_0(\tau, x)) d\tau \quad (A4.2.6)$$

and, using (A4.2.3) and the fact that $w_0(0, x) = 0$,

$$|w_\epsilon(t, x)| \leq \gamma(t) t e^{-\epsilon t} + \epsilon \int_0^t e^{-\epsilon(t-\tau)} (t-\tau) \gamma(t-\tau) d\tau \quad (A4.2.7)$$

Consequently,

$$|\epsilon w_\epsilon(t, x)| \leq \sup_{t' \geq 0} \gamma \left[\frac{t'}{\epsilon} \right] t' e^{-t'} + \int_0^\infty \gamma \left[\frac{\tau'}{\epsilon} \right] \tau' e^{-\tau'} d\tau' \quad (A4.2.8)$$

Since, for some β , $|d(t, x)| \leq \beta$, we also have that $\gamma(t) \leq \beta$. Note that, for all $t' \geq 0$, $t' e^{-t'} \leq e^{-1}$, and $t' e^{-t'} \leq t'$, so that

$$\begin{aligned}
|\epsilon w_\epsilon(t, x)| &\leq \sup_{t' \in [0, \sqrt{\epsilon}]} \left[\gamma \left[\frac{t'}{\epsilon} \right] t' e^{-t'} \right] + \sup_{t' \geq \sqrt{\epsilon}} \left[\gamma \left[\frac{t'}{\epsilon} \right] t' e^{-t'} \right] \\
&\quad + \int_0^{\sqrt{\epsilon}} \gamma \left[\frac{\tau'}{\epsilon} \right] \tau' e^{-\tau'} d\tau' + \int_{\sqrt{\epsilon}}^\infty \gamma \left[\frac{\tau'}{\epsilon} \right] \tau' e^{-\tau'} d\tau' \quad (A4.2.9)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
|\epsilon w_\epsilon(t, x)| &\leq \beta \sqrt{\epsilon} + \gamma \left[\frac{1}{\sqrt{\epsilon}} \right] e^{-1} + \beta \frac{\epsilon}{2} + \gamma \left[\frac{1}{\sqrt{\epsilon}} \right] (1 + \sqrt{\epsilon}) e^{-\sqrt{\epsilon}} \\
&:= \xi(\epsilon) \quad (A4.2.10)
\end{aligned}$$

From the assumption on γ , it follows that $\xi(\epsilon) \in K$. From (A4.2.1)

$$\frac{\partial w_\epsilon(t, x)}{\partial t} - d(t, x) = -\epsilon w_\epsilon(t, x) \quad (A4.2.11)$$

so that the first part of the lemma is verified.

If $\gamma(T) = a/T^r$, then the right-hand side of (A4.2.8) can be computed explicitly

$$\sup_{t' \geq 0} a \epsilon'(t')^{1-r} e^{-t'} = a \epsilon^r (1-r)^{1-r} e^{r-1} \leq a \epsilon^r \quad (\text{A4.2.12})$$

and, with Γ denoting the standard gamma function,

$$\int_0^{\infty} a \epsilon'(t')^{1-r} e^{-t'} dt' = a \epsilon^r \Gamma(2-r) \leq a \epsilon^r \quad (\text{A4.2.13})$$

Defining $\xi(\epsilon) = 2a\epsilon^r$, the second part of the lemma is verified. \square

Proof of Lemma 4.2.2

Define $w_\epsilon(t, x)$ as in lemma 4.2.1. Consequently,

$$\begin{aligned} \frac{\partial w_\epsilon(t, x)}{\partial x} &= \frac{\partial}{\partial x} \left[\int_0^t d(\tau, x) e^{-\epsilon(t-\tau)} d\tau \right] \\ &= \int_0^t \left[\frac{\partial}{\partial x} d(\tau, x) \right] e^{-\epsilon(t-\tau)} d\tau \end{aligned} \quad (\text{A4.2.14})$$

Since $\frac{\partial d(t, x)}{\partial x}$ is zero mean, and is bounded, lemma 4.2.1 can be applied to $\frac{\partial d(t, x)}{\partial x}$, and inequality (4.2.6) of lemma 4.2.1 becomes inequality (4.2.10) of lemma 4.2.2. Note that since $\frac{\partial d(t, x)}{\partial x}$ is bounded, and $d(t, 0) = 0$ for all $t \geq 0$, $d(t, x)$ is Lipschitz.

Since $d(t, x)$ is zero mean, with convergence function $\gamma(T)|x|$, the proof of lemma 4.2.1 can be extended, with an additional factor $|x|$. This leads directly to (4.2.8) and (4.2.9) (although the function $\xi(\epsilon)$ may be different from that obtained with $\frac{\partial d(t, x)}{\partial x}$, these functions can be replaced by a single $\xi(\epsilon)$). \square

Proof of Lemma 4.2.3

The proof proceeds in two steps.

(a) For ϵ sufficiently small, and for t fixed, the transformation is a homeomorphism.

Apply lemma 4.2.2, and let ϵ_1 such that $\xi(\epsilon_1) < 1$. Let $\epsilon \leq \epsilon_1$. Given $z \in B_h$, the corresponding x such that

$$x = z + \epsilon w_\epsilon(t, z) \quad (\text{A4.2.15})$$

may not belong to B_h . Similarly, given $x \in B_h$, the solution z of (A4.2.15) may not exist in B_h . However, for any x, z satisfying (A4.2.15), inequality (4.2.8) implies (4.2.16) and

$$(1 - \xi(\epsilon))|z| \leq |x| \leq (1 + \xi(\epsilon))|z| \quad (\text{A4.2.16})$$

Define

$$h'(\epsilon) = \min \left[h(1 - \xi(\epsilon)), \frac{h}{1 + \xi(\epsilon)} \right] = h(1 - \xi(\epsilon)) \quad (\text{A4.2.17})$$

and note that $h'(\epsilon) \rightarrow h$ as $\epsilon \rightarrow 0$.

We now show that

- for all $z \in B_h$, there exists a unique $x \in B_h$ such that (A4.2.15) is satisfied,
- for all $x \in B_h$, there exists a unique $z \in B_h$ such that (A4.2.15) is satisfied.

In both cases, $|x - z| \leq \xi(\epsilon)h$.

The first part follows directly from (A4.2.15), (A4.2.16). The fact that $|x - z| \leq \xi(\epsilon)h$ also follows from (A4.2.15), (4.2.8) and implies that, if a solution z exists to (A4.2.15), it must lie in the closed ball U of radius $\xi(\epsilon)h$ around x . It can be checked, using (4.2.10), that the mapping $F_x(z) = x - \epsilon w_\epsilon(t, z)$ is a contraction mapping in U , provided that $\xi(\epsilon) < 1$. Consequently, F has a unique fixed point z in U . This solution is also a solution of (A4.2.15), and since it is unique in U , it is also unique in B_h (and actually in R^n). For $x \in B_h$, but outside B_h , there is no guarantee that a solution z exists in B_h , but if it exists, it is again unique in B_h . Consequently, the map $x \rightarrow z$ defined by (A4.2.15) is well defined. From the smoothness of $w_\epsilon(t, z)$ with respect to z , it follows that the map is a homeomorphism.

(b) The transformation of variable leads to the differential equation (4.2.17)

Applying (A4.2.15) to the system (4.2.1)

$$\begin{aligned} \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z} \right] \dot{z} &= \epsilon f_{av}(z) + \epsilon \left[f(t, z, 0) - f_{av}(z) - \frac{\partial w_\epsilon}{\partial t} \right] \\ &+ \epsilon \left[f(t, z + \epsilon w_\epsilon, \epsilon) - f(t, z, \epsilon) \right] \end{aligned}$$

$$\begin{aligned}
& + \epsilon \left[f(t, z, \epsilon) - f(t, z, 0) \right] \\
& := \epsilon f_{av}(z) + \epsilon p'(t, z, \epsilon) \quad (A4.2.18)
\end{aligned}$$

where, using the assumptions, and the results of lemma 4.2.2

$$|p'(t, z, \epsilon)| \leq \xi(\epsilon)|z| + \xi(\epsilon)l_1|z| + \epsilon l_2|z| \quad (A4.2.19)$$

For $\epsilon \leq \epsilon_1$, (4.2.10) implies that $\left[I + \epsilon \frac{\partial w_\epsilon}{\partial z} \right]$ has a bounded inverse for all $t \geq 0$, $z \in B_h$. Consequently, z satisfies the differential equation

$$\begin{aligned}
\dot{z} &= \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z} \right]^{-1} \left[\epsilon f_{av}(z) + \epsilon p'(t, z, \epsilon) \right] \\
&= \epsilon f_{av}(z) + \epsilon p(t, z, \epsilon) \quad z(0) = x_0 \quad (A4.2.20)
\end{aligned}$$

where

$$p(t, z, \epsilon) = \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z} \right]^{-1} \left[p'(t, z, \epsilon) - \epsilon \frac{\partial w_\epsilon}{\partial z} f_{av}(z) \right] \quad (A4.2.21)$$

and

$$\begin{aligned}
|p(t, z, \epsilon)| &\leq \frac{1}{1 - \xi(\epsilon_1)} \left[\xi(\epsilon) + \xi(\epsilon)l_1 + \epsilon l_2 + \xi(\epsilon)l_{av} \right] |z| \\
&:= \psi(\epsilon)|z| \quad (A4.2.22)
\end{aligned}$$

for all $t \geq 0$, $\epsilon \leq \epsilon_1$, $z \in B_h$. \square

Proof of Lemma 4.4.1

The proof is similar to the proof of lemma 4.2.3. We consider the transformation of variable

$$x = z + \epsilon w_\epsilon(t, z) \quad (A4.4.1)$$

with $\epsilon \leq \epsilon_1$, such that $\xi(\epsilon_1) < 1$. (4.4.1) becomes

$$\begin{aligned}
\dot{z} &= \left[I + \epsilon \frac{\partial w_\epsilon}{\partial z} \right]^{-1} \left[\epsilon f_{av}(z) + \left[f(t, z, 0) - f_{av}(z) - \frac{\partial w_\epsilon}{\partial t} \right] \right. \\
&\quad \left. + \left[f(t, z + \epsilon w_\epsilon, 0) - f(t, z, 0) \right] \right]
\end{aligned}$$

$$+ \left[f(t, z + \epsilon w_\epsilon, y) - f(t, z + \epsilon w_\epsilon, 0) \right] \} \quad (A4.4.2)$$

or

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p_1(t, z, \epsilon) + \epsilon p_2(t, z, y, \epsilon) \quad z(0) = x_0 \quad (A4.4.3)$$

where

$$\begin{aligned}
|p_1(t, z, \epsilon)| &\leq \frac{1}{1 - \xi(\epsilon_1)} \left[\xi(\epsilon)l_{av} + \xi(\epsilon) + \xi(\epsilon)l_1 \right] |z| \\
&:= \xi(\epsilon)k_1|z| \quad (A4.4.4)
\end{aligned}$$

and

$$|p_2(t, z, y, \epsilon)| \leq \frac{1}{1 - \xi(\epsilon_1)} l_2|y| := k_2|y| \quad (A4.4.5)$$

\square

Proof of Theorem 4.4.2

The proof assumes that for all $t \in [0, T/\epsilon]$, the solutions $x(t)$, $y(t)$, and $z(t)$ (to be defined) remain in B_h . Since this is not guaranteed a priori, the steps of the proof are only valid for as long as the condition is verified. By assumption, $x_{av}(t) \in B_{h'}$, with $h' < h$. We will show that by letting ϵ and h_0 sufficiently small, we can let $x(t)$ be arbitrarily close to $x_{av}(t)$ and $y(t)$ arbitrarily small. It then follows, from a contradiction argument, that $x(t)$, $y(t) \in B_h$ for all $t \in [0, T/\epsilon]$, provided that ϵ and h_0 are sufficiently small.

Using lemma 4.4.1, we transform the original system (4.4.1), (4.4.2) into the system (4.4.11), (4.4.2). A bound on the error $|z(t) - x_{av}(t)|$ can be calculated by integrating the difference (4.4.11)–(4.4.4), and by using (4.4.7) and (4.4.12)

$$\begin{aligned}
|z(t) - x_{av}(t)| &\leq \epsilon l_{av} \int_0^t |z(\tau) - x_{av}(\tau)| d\tau + \epsilon \xi(\epsilon) k_1 \int_0^t |z(\tau)| d\tau \\
&\quad + \epsilon k_2 \int_0^t |y(\tau)| d\tau \quad (A4.4.6)
\end{aligned}$$

Bound on $|y(t)|$

To obtain a bound on $|y(t)|$, we divide the interval $[0, T/\epsilon]$ in intervals $[t_i, t_{i+1}]$ of length ΔT (the last interval may be of smaller length, and ΔT will be defined later). The differential equation for y is

$$\dot{y} = A(x)y + \epsilon g(t, x, y) \quad (\text{A4.4.7})$$

and is rewritten on the time interval $[t_i, t_{i+1}]$ as follows

$$\dot{y} = A_{x_i}y + \epsilon g(t, x, y) + (A_{x_t} - A_{x_i})y \quad (\text{A4.4.8})$$

where $A_{x_t} = A(x(t))$, and $A_{x_i} = A(x(t_i))$, so that the solution $y(t)$, for $t \in [t_i, t_{i+1}]$, is given by

$$\begin{aligned} y(t) = & e^{A_{x_i}(t-t_i)} y_i + \epsilon \int_{t_i}^t e^{A_{x_i}(t-\tau)} g(\tau, x, y) d\tau \\ & + \int_{t_i}^t e^{A_{x_i}(t-\tau)} (A_{x_\tau} - A_{x_i}) y(\tau) d\tau \end{aligned} \quad (\text{A4.4.9})$$

where $y_i = y(t_i)$. From the assumptions, it follows that

$$\|A_{x_\tau} - A_{x_i}\| \leq k_a |\dot{x}| (\tau - t_i) \leq \epsilon(l_1 + l_2) h k_a \Delta \quad (\text{A4.4.10})$$

and, using the uniform exponential stability assumption on $A(x)$

$$|y(t)| \leq m |y_i| e^{-\lambda(t-t_i)} + \epsilon \frac{m}{\lambda} h \left[(l_3 + l_4) + (l_1 + l_2) k_a \Delta T \right] \quad (\text{A4.4.11})$$

Let the last term in (A4.4.11) be denoted by ϵk_b , and use (A4.4.11) as a recursion formula for y_i , so that

$$|y_i| \leq \left[m e^{-\lambda \Delta T} \right]^i |y_0| + \epsilon k_b \sum_{j=0}^{i-1} \left[m e^{-\lambda \Delta T} \right]^j \quad (\text{A4.4.12})$$

Choose ΔT sufficiently large that

$$m e^{-\lambda \Delta T} \leq e^{-\lambda \Delta T/2} \quad \text{or} \quad \Delta T \geq \frac{2}{\lambda} \ln m \quad (\text{A4.4.13})$$

It follows that

$$\sum_{j=0}^{i-1} \left[m e^{-\lambda \Delta T} \right]^j \leq \sum_{j=0}^{\infty} \left[e^{-\lambda \Delta T/2} \right]^j = \frac{1}{1 - e^{-\lambda \Delta T/2}} \quad (\text{A4.4.14})$$

Combining (A4.4.12)–(A4.4.14) and using the assumption $y_0 \in B_{h_0}$,

$$|y_i| \leq e^{-\lambda \Delta T i/2} h_0 + \frac{\epsilon k_b}{1 - e^{-\lambda \Delta T/2}} := e^{-\lambda t_i/2} h_0 + \epsilon k_c \quad (\text{A4.4.15})$$

Using this result in (A4.4.11), it follows that for all $t \in [t_i, t_{i+1}]$

$$|y(t)| \leq m e^{-\lambda t_i/2} h_0 e^{-\lambda(t-t_i)} + m \epsilon k_c e^{-\lambda(t-t_i)} + \epsilon k_b$$

$$\leq m h_0 e^{-\lambda t/2} + \epsilon (m k_c + k_b) \quad (\text{A4.4.16})$$

Since the last inequality does not depend on i , it gives a bound on $|y(t)|$ for all $t \in [0, T/\epsilon]$.

Bound on $z(t) - x_{av}(t)$

We now return to (A4.4.6), and to the approximation error, using the bound on $|y(t)|$

$$\begin{aligned} |z(t) - x_{av}(t)| \leq & \epsilon l_{av} \int_0^t |z(\tau) - x_{av}(\tau)| d\tau + \epsilon \xi(\epsilon) k_1 \int_0^t h d\tau \\ & + \epsilon k_2 \int_0^t \left[m h_0 e^{-\lambda \tau/2} + \epsilon (m k_c + k_b) \right] d\tau \end{aligned} \quad (\text{A4.4.17})$$

so that, using the Bellman-Gronwall lemma (lemma 1.4.2)

$$\begin{aligned} |z(t) - x_{av}(t)| & \leq \int_0^t \left[\xi(\epsilon) k_1 h + k_2 m h_0 e^{-\lambda \tau/2} + k_2 \epsilon (m k_c + k_b) \right] \epsilon e^{\epsilon l_{av}(t-\tau)} d\tau \\ & \leq (\epsilon + \xi(\epsilon)) \left[k_1 h + \frac{k_2 m h_0 l_{av}}{\lambda/2 + \epsilon l_{av}} + k_2 (m k_c + k_b) \right] \frac{e^{\epsilon l_{av} T}}{l_{av}} \\ & := \psi(\epsilon) a_T \end{aligned} \quad (\text{A4.4.18})$$

and, using (4.4.10)

$$|x(t) - x_{av}(t)| \leq \psi(\epsilon) b_T \quad (\text{A4.4.19})$$

for some b_T .

Assumptions

We assumed in the proof that all signals remained in B_h . By assumption, $x_{av}(t) \in B_{h'}$, for some $h' < h$. Let h_0 , and ϵ_T be sufficiently small so that, for all $\epsilon \leq \epsilon_T \leq \epsilon_1$, we have that $m h_0 + \epsilon (m k_c + k_b) \leq h$ (cf. (A4.4.16)), and that $\psi(\epsilon) b_T \leq h - h'$ (cf. (4.4.27)). It follows, from a simple contradiction argument, that the solutions $x(t)$, $y(t)$ and $z(t)$ remain in B_h for all $t \in [0, T/\epsilon]$, so that all steps of the proof are valid, and (A4.4.19) is in fact satisfied over the whole time interval. \square

Proof of Theorem 4.4.3

The proof relies on the converse theorem of Lyapunov for exponentially stable systems (theorem 1.4.3). Under the hypotheses, there exists a function $v(x_{av}) : \mathbb{R}^n \rightarrow \mathbb{R}_+$, and strictly positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that, for all $x_{av} \in B_h$,

$$\alpha_1 |x_{av}|^2 \leq v(x_{av}) \leq \alpha_2 |x_{av}|^2 \quad (\text{A4.4.20})$$

$$\dot{v}(x_{av}) \Big|_{(4.4.4)} \leq -\epsilon \alpha_3 |x_{av}|^2 \quad (\text{A4.4.21})$$

$$\left| \frac{\partial v}{\partial x_{av}} \right| \leq \alpha_4 |x_{av}| \quad (\text{A4.4.22})$$

The derivative in (A4.4.21) is to be taken along the trajectories of the averaged system (4.4.4).

We now study the stability of the original system (4.4.1), (4.4.2), through the transformed system (4.4.11), (4.4.2), where $x(z)$ is defined in (4.4.9). Consider the following Lyapunov function

$$v_1(z, y) = v(z) + \frac{\alpha_2}{p_2} y^T P(x(z)) y \quad (\text{A4.4.23})$$

where $P(x)$, p_2 are defined in the comments after the definition of uniform exponential stability of $A(x)$. Defining $\alpha'_1 = \min(\alpha_1, \frac{\alpha_2}{p_2} p_1)$, it follows that

$$\alpha'_1 (|z|^2 + |y|^2) \leq v_1(z, y) \leq \alpha_2 (|z|^2 + |y|^2) \quad (\text{A4.4.24})$$

The derivative of v_1 along the trajectories of (4.4.11)–(4.4.2) can be bounded, using the foregoing inequalities

$$\begin{aligned} \dot{v}_1(z, y) &\leq -\epsilon \alpha_3 |z|^2 + \epsilon \xi(\epsilon) k_1 \alpha_4 |z|^2 + \epsilon k_2 \alpha_4 |z| |y| \\ &\quad + \frac{\alpha_2}{p_2} \left\| \frac{\partial P(x)}{\partial x} \right\| \left\| \frac{\partial x}{\partial z} \right\| |z| |y|^2 \\ &\quad - \frac{\alpha_2}{p_2} q_1 |y|^2 + 4\epsilon l_3 \alpha_2 |z| |y| + 2\epsilon l_4 \alpha_2 |y|^2 \end{aligned} \quad (\text{A4.4.25})$$

for $\epsilon \leq \epsilon_1$ (so that the transformation $x \rightarrow z$ is well defined and $|x| \leq 2|z|$). We now calculate bounds on the terms in (A4.4.25).

Bound on $\|\partial P/\partial x\|$

Note that $P(x)$ can be defined by

$$P(x) = \int_0^\infty e^{A^T(x)t} Q e^{A(x)t} dt \quad (\text{A4.4.26})$$

so that

$$\begin{aligned} \frac{\partial P(x)}{\partial x_i} &= \int_0^\infty \left\{ \left[\frac{\partial}{\partial x_i} e^{A^T(x)t} \right] Q e^{A(x)t} \right. \\ &\quad \left. + e^{A^T(x)t} Q \left[\frac{\partial}{\partial x_i} e^{A(x)t} \right] \right\} dt \end{aligned} \quad (\text{A4.4.27})$$

The partial derivatives in parentheses solve the differential equation

$$\frac{d}{dt} \left[\frac{\partial}{\partial x_i} e^{A(x)t} \right] = A(x) \left[\frac{\partial}{\partial x_i} e^{A(x)t} \right] + \frac{\partial A(x)}{\partial x_i} e^{A(x)t} \quad (\text{A4.4.28})$$

with zero initial conditions, so that

$$\frac{\partial}{\partial x_i} e^{A(x)t} = \int_0^t e^{A(x)(t-\tau)} \frac{\partial A(x)}{\partial x_i} e^{A(x)\tau} d\tau \quad (\text{A4.4.29})$$

From the boundedness of $\frac{\partial A(x)}{\partial x_i}$, and from the exponential stability of $A(x)$, it follows that

$$\left\| \frac{\partial}{\partial x} e^{A(x)t} \right\| \leq m^2 k_a t e^{-\lambda t} \quad (\text{A4.4.30})$$

With (A4.4.27), this implies that $\|\partial P(x)/\partial x\|$ is bounded by some $k_p \geq 0$.

Bound on $\|\partial x/\partial z\|$ and $|z|$

On the other hand, using (4.4.9), (4.2.8) and (4.4.12)

$$\left\| \frac{\partial x}{\partial z} \right\| < 1 + \xi(\epsilon) < 2$$

$$\text{and } |z| \leq \epsilon h(l_{av} + \xi(\epsilon)k_1 + k_2) \quad (\text{A4.4.31})$$

Using these results in (A4.4.25), and noting the fact that, for all y , $z \in \mathbb{R}$

$$\epsilon |z| |y| \leq \frac{1}{2} (\epsilon^{4/3} |z|^2 + \epsilon^{2/3} |y|^2) \quad (\text{A4.4.32})$$

it follows that

$$\begin{aligned} \dot{v}_1(z, y) &\leq -\epsilon \left[\alpha_3 - \xi(\epsilon) k_1 \alpha_4 - \epsilon^{1/3} \frac{k_2 \alpha_4}{2} - 2 \epsilon^{1/3} l_3 \alpha_2 \right] |z|^2 \\ &\quad - \left[\frac{\alpha_2}{p_2} q_1 - 2 \epsilon l_4 \alpha_2 - \epsilon^{2/3} \frac{k_2 \alpha_4}{2} - 2 \epsilon^{2/3} l_3 \alpha_2 \right. \\ &\quad \left. + 2 \epsilon \frac{\alpha_2}{p_2} k_p h \left[l_{av} + \xi(\epsilon) k_1 + k_2 \right] \right] |y|^2 \\ &:= -2 \epsilon \alpha_2 \alpha(\epsilon) |z|^2 - q(\epsilon) |y|^2 \end{aligned} \quad (\text{A4.4.33})$$

Note that, with this definition, $\alpha(\epsilon) \rightarrow \frac{1}{2} \frac{\alpha_3}{\alpha_2}$ as $\epsilon \rightarrow 0$, while

$$q(\epsilon) \rightarrow \frac{\alpha_2}{p_2} q_1.$$

Let $\epsilon \leq \epsilon_1$ be sufficiently small that $\alpha(\epsilon) > 0$ and $2 \epsilon \alpha_2 \alpha(\epsilon) \leq q(\epsilon)$. Then

$$\dot{v}_1(z, y) \leq -2 \epsilon \alpha(\epsilon) v_1(z, y) \quad (\text{A4.4.34})$$

so that the z, y system is exponentially stable with rate of convergence $\epsilon \alpha(\epsilon)$ (v_1 being bounded above and below by the square of the norm of the state). The same conclusion holds for the x, y system, given the transformation (4.4.9), with (4.4.10). Also, for ϵ, h_0 sufficiently small, all signals are actually guaranteed to remain in B_h so that all assumptions are valid. \square

Auxiliary Lemmas for Section 6.2

Lemma A6.2.1

Consider the least squares identification algorithm described by (6.2.8), (6.2.9) with the sequence of resetting times $\{0, t_1, t_2, \dots\}$, that is

$$\dot{\phi} = -P w w^T \phi \quad (\text{A6.2.1})$$

$$\frac{d}{dt} (P^{-1}) = w w^T \quad t \neq t_i \quad (\text{A6.2.2})$$

$$P^{-1}(t_i^+) = k_0 I \quad t_i = 0, t_1, t_2, \dots \quad (\text{A6.2.3})$$

If w satisfies

$$\int_{t_i}^{t_{i+1}} w w^T dt \geq \alpha_1 I \quad \text{for all } t_i \quad (\text{A6.2.4})$$

Then

$$|\phi(t_i)| \leq \left[\frac{k_0}{k_0 + \alpha_1} \right]^i |\phi(0)| \quad (\text{A6.2.5})$$

Proof of Lemma A6.2.1

Note that for $t \neq 0, t_1, t_2, \dots$

$$\frac{d}{dt} (P^{-1} \phi) = 0 \quad (\text{A6.2.6})$$

Thus

$$P^{-1}(t_i^-) \phi(t_i) = P^{-1}(t_{i-1}^+) \phi(t_{i-1}) \quad (\text{A6.2.7})$$

so that

$$\phi(t_i) = k_0 P(t_i^-) \phi(t_{i-1}) \quad (\text{A6.2.8})$$

and, with (A6.2.4)

$$|\phi(t_i)| \leq \left[\frac{k_0}{k_0 + \alpha_1} \right] |\phi(t_{i-1})| \quad (\text{A6.2.9})$$

Recursion on (A6.2.9) yields (A6.2.5).

Comments

If $\alpha_1 = 0$, the lemma shows that $\phi(t_i)$ is bounded. If $\alpha_1 > 0$ and the sequence t_i is infinite, $\phi(t_i) \rightarrow 0$ as $i \rightarrow \infty$. Further, if the intervals $t_{i+1} - t_i$ are bounded, then $\phi(t_i) \rightarrow 0$ exponentially. \square

Lemma A6.2.2

Consider the following linear systems

$$\dot{z}_0 = A z_0 + b r \quad (\text{A6.2.10})$$

$$\dot{z} = (A + \Delta A(t)) z + (b + \Delta b(t)) r \quad (\text{A6.2.11})$$

with A stable and $\Delta A, \Delta b$ both bounded and converging to zero as

$t \rightarrow \infty$.

If the input r is bounded

Then given $\epsilon > 0$, there exists $k > 0$ (independent of ϵ) and a $T(\epsilon)$ such that

$$|z(t) - z_0(t)| \leq \epsilon k \quad \text{for all } t \geq T \quad (\text{A6.2.12})$$

Proof of Lemma A6.2.2

From lemma 6.2.1, it follows that $A + \Delta A(t)$ is asymptotically stable and that there exists T_1 such that the state transition matrix of $A + \Delta A(t)$ satisfies

$$\|\phi(t, \tau)\| \leq m (\exp(-\alpha(t - \tau))) \quad (\text{A6.2.13})$$

for some $m, \alpha > 0$ and $t \geq \tau > T_1$. Using this estimate, it is easy to show that $z(t)$ is bounded. Now, defining the error $e(t) := z(t) - z_0(t)$, we have that

$$\dot{e} = Ae + \Delta A z + \Delta b r \quad (\text{A6.2.14})$$

For T sufficiently large, ΔA and Δb are arbitrarily small, so that e may be showed to satisfy (A6.2.12). \square

Lemma A6.2.3 Solution of the Pole Placement Equation

Consider two coprime polynomials: \hat{d}_p monic of order n , and \hat{n}_p monic of order $\leq n - 1$. Let k_p be a real number.

Then given an arbitrary polynomial $\hat{d}_{cl}(s)$ of order $2n - 1$, there exist unique polynomials \hat{n}_c and \hat{d}_c of order at most $n - 1$ so that

$$\hat{n}_c k_p \hat{n}_p + \hat{d}_c \hat{d}_p = \hat{d}_{cl} \quad (\text{A6.2.15})$$

Proof of Lemma A6.2.3

Since $k_p \hat{n}_p$ and \hat{d}_p are coprime and of order $n - 1, n$, respectively, there exist polynomials \hat{u}, \hat{v} of degree at most $n - 1$, respectively so that

$$\hat{u} k_p \hat{n}_p + \hat{v} \hat{d}_p = 1 \quad (\text{A6.2.16})$$

Thus, we see that

$$\hat{u} \hat{d}_{cl} k_p \hat{n}_p + \hat{v} \hat{d}_{cl} \hat{d}_p = \hat{d}_{cl} \quad (\text{A6.2.17})$$

Further, we may modify (A6.2.17) to

$$(\hat{u} \hat{d}_{cl} - \hat{q} \hat{d}_p) k_p \hat{n}_p + (\hat{v} \hat{d}_{cl} + \hat{q} k_p \hat{n}_p) \hat{d}_p = \hat{d}_{cl} \quad (\text{A6.2.18})$$

for an arbitrary polynomial \hat{q} . Let

$$\begin{aligned} \hat{n}_c &:= \hat{u} \hat{d}_{cl} - \hat{q} \hat{d}_p \\ \hat{d}_c &:= \hat{v} \hat{d}_{cl} + \hat{q} k_p \hat{n}_p \end{aligned} \quad (\text{A6.2.19})$$

Since \hat{d}_p is of order n , we may choose \hat{q} so that \hat{n}_c is of order $\leq n - 1$ (for instance, as the quotient obtained by dividing $\hat{u} \hat{d}_{cl}$ by \hat{d}_p). Then, \hat{d}_c is constrained to be of order $\leq n - 1$, since the other two polynomials in (A6.2.18), that is $\hat{n}_c k_p \hat{n}_p$ and \hat{d}_{cl} , are of order $\leq 2n - 1$.

It is useful to note that if

$$\begin{aligned} \hat{d}_{cl} &= d_{2n} s^{2n-1} + \dots + d_1 \\ \hat{d}_p &= s^n + \beta_n s^{n-1} + \dots + \beta_1 \\ k_p \hat{n}_p &= \alpha_n s^{n-1} + \dots + \alpha_1 \\ \hat{n}_c &= a_n s^{n-1} + \dots + a_1 \\ \hat{d}_c &= b_n s^{n-1} + \dots + b_1 \end{aligned} \quad (\text{A6.2.20})$$

then the linear equation relating the coefficients of \hat{n}_c, \hat{d}_c to those of \hat{d}_{cl} is

$$\begin{bmatrix} \alpha_1 & 0 & \dots & 0 & 0 & \beta_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \alpha_1 & \dots & 0 & 0 & \beta_2 & \beta_1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \alpha_{n-1} & \alpha_{n-2} & \dots & \alpha_1 & 0 & \beta_{n-1} & \beta_{n-2} & \dots & \beta_1 & 0 \\ \alpha_n & \alpha_{n-1} & \dots & \alpha_2 & \alpha_1 & \beta_n & \beta_{n-1} & \dots & \beta_2 & \beta_1 \\ 0 & \alpha_n & \dots & \alpha_3 & \alpha_2 & 1 & \beta_n & \dots & \beta_3 & \beta_2 \\ 0 & 0 & \dots & \alpha_4 & \alpha_3 & 0 & 1 & \dots & \beta_4 & \beta_3 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \alpha_n & 0 & 0 & \dots & 1 & \beta_n \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_{n-1} \\ a_n \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_{n-1} \\ d_n \\ d_{n+1} \\ d_{n+2} \\ \cdot \\ \cdot \\ d_{2n-1} \\ d_{2n} \end{bmatrix} \quad (\text{A6.2.21})$$

(A6.2.21) is of the form $A(\theta^*)\theta_c^* = d$ where θ^* is the nominal plant parameter, and θ_c^* the nominal controller parameter. \square

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