as desired.

d) Implement the appropriate start-up features for the algorithm using prior knowledge about the plant to choose initial parameter values and include "safety nets" to cover start-up, shut-down and transitioning between various modes of operation of the overall controller.

The guidelines given in this chapter are for the most part conceptual: in applications, questions of numerical conditioning of signals, sampling intervals (for digital implementations), anti-aliasing filters (for digital implementations), controller-architecture featuring several levels of interruptability, resetting, and so on are important. Even with a considerable wealth of theory and analysis of the algorithms, the difference an adaptive controller makes in a given application is chiefly due to the art of the designer!

CHAPTER 6

ADVANCED TOPICS IN IDENTIFICATION AND ADAPTIVE CONTROL

6.1 USE OF PRIOR INFORMATION

6.1.1 Identification of Partially Known Systems

We consider in this section the problem of identifying partially known single-input single-output (SISO) transfer functions of the form

$$\hat{P}(s) = \frac{\hat{N}_0(s) + \sum_{i=1}^{m} \alpha_i \hat{N}_i(s)}{\hat{D}_0(s) - \sum_{j=1}^{n} \beta_j \hat{D}_j(s)} \tag{6.1.1}$$

where $\hat{N}_i$ and $\hat{D}_j$ are known, proper, stable rational transfer functions and α_i, β_j are unknown, real parameters. The identification problem is to identify α_i, β_j from input-output measurements of the system. The problem was recently addressed by Clary [1984], Dasgupta [1984], and Bai and Sastry [1986].

The representation (6.1.1) is general enough to model several kinds of "partially known" systems.

Examples

a) *Network functions of RLC circuits with some elements unknown.* Consider for example the circuit of Figure 6.1, with the resistor R unknown (the circuit is drawn as a two port to exhibit the unknown resistance).

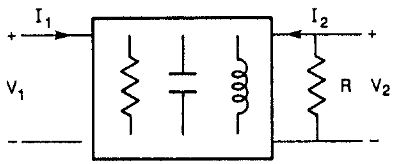

Figure 6.1 Two Port with Unknown Resistance R

If the short circuit admittance matrix of the two port in Figure 6.1 is

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} y_{11}(s) & y_{12}(s) \\ y_{21}(s) & y_{22}(s) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{6.1.2}$$

then a simple calculation yields the admittance function

$$\frac{i_1}{v_1} = \frac{y_{11} + R(y_{11}y_{22} - y_{12}y_{21})}{1 + Ry_{22}} \tag{6.1.3}$$

which is of the form of (6.1.1). Circuits with more than one unknown element can be drawn as multiports to show that the admittance function is of the form of (6.1.1).

b) *Interconnections of several known systems with unknown interconnection gains.* A simple example of this is shown in Figure 6.2, with a plant $\hat{P}(s)$ known, and a feedback gain k unknown.

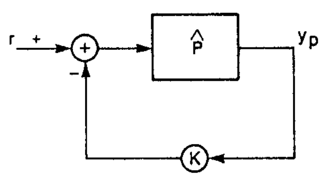

Figure 6.2 Plant with Unknown Feedback Gain

The closed-loop transfer function, namely $\hat{P} / 1 + k\hat{P}$ is of the form of (6.1.1) if $\hat{P}$ is stable. If $\hat{P}$ is unstable, then by writing $\hat{P} = \hat{N}_p / \hat{D}_p$ as the ratio of two proper stable rational transfer functions, the closed-loop transfer function is $\hat{N}_p / \hat{D}_p + k\hat{N}_p$, which is of the form (6.1.1).

c) *Classical transfer function models,* that is, plants of the form studied in Chapter 2

$$\hat{P}(s) = \frac{\alpha_m s^{m-1} + \cdots + \alpha_1}{s^n + \beta_n s^{n-1} + \cdots + \beta_1} \tag{6.1.4}$$

with $m \le n$ and α_i, β_j unknown, can be stated in terms of the set up of (6.1.1) by choosing

$$\begin{aligned} \hat{N}_0(s) &= 0 \\ \hat{D}_0(s) &= s^n / \hat{\lambda}(s) \\ \hat{N}_i(s) &= s^{i-1} / \hat{\lambda}(s) \qquad i = 1, \ldots, m \\ \hat{D}_j(s) &= -s^{j-1} / \hat{\lambda}(s) \qquad j = 1, \ldots, n \end{aligned} \tag{6.1.5}$$

where $\hat{\lambda}(s)$ is (any) Hurwitz polynomial of order n.

d) Systems with some known poles and zeros. Consider the system of Figure 6.3, with unknown plant, but known actuator and sensor dynamics (with transfer functions $\hat{P}_a(s)$ and $\hat{P}_s(s)$ respectively).

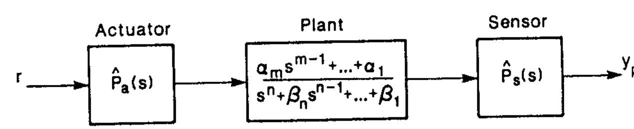

Figure 6.3 Unknown Plant with Known Actuator and Sensor Dynamics

The overall transfer function is written as

$$\hat{P}(s) = \hat{P}_s(s) \cdot \frac{\sum_{i=1}^{m} \alpha_i s^{i-1}}{s^n + \sum_{j=1}^{n} \beta_j s^{j-1}} \cdot \hat{P}_a(s) \tag{6.1.6}$$

which is of the form (6.1.1) by choosing, as above

$$\begin{aligned} \hat{N}_0(s) &= 0 \\ \hat{D}_0(s) &= s^n / \hat{\lambda}(s) \\ \hat{N}_i(s) &= s^{i-1} \hat{P}_a(s) \hat{P}_s(s) / \hat{\lambda}(s) \qquad i = 1, \ldots, m \\ \hat{D}_j(s) &= -s^{j-1} / \hat{\lambda}(s) \qquad j = 1, \ldots, n \end{aligned} \tag{6.1.7}$$

where $\hat{\lambda}(s)$ is (any) Hurwitz polynomial of order n.

Identification Scheme

We now return to the general system described by (6.1.1). The identification problem is to determine the unknown parameters α_i, β_j from input-output measurements. One could, of course, neglect the prior information embedded in the form of the transfer function (6.1.1) and identify the whole transfer function using one of the procedures of Chapter 2. However, usage of the particular structure embodied in (6.1.1) will result in the identification of fewer unknown parameters, with a reduction of computational requirements and usually faster convergence properties.

Let $\hat{r}(s)$, $\hat{y}_p(s)$ denote the input and output of the plant. Using (6.1.1)

$$\hat{D}_0 \hat{y}_p - \hat{N}_0 \hat{r} = \sum_{j=1}^{n} \beta_j \hat{D}_j \hat{y}_p + \sum_{i=1}^{m} \alpha_i \hat{N}_i \hat{r} \tag{6.1.8}$$

Defining the signals

$$\begin{aligned} \hat{z}_p &:= \hat{D}_0 \hat{y}_p - \hat{N}_0 \hat{r} \\ \hat{w}_i &:= \hat{N}_i \hat{r} \qquad i = 1, \ldots, m \\ \hat{w}_{m+j} &:= \hat{D}_j \hat{y}_p \qquad j = 1, \ldots, n \end{aligned} \tag{6.1.9}$$

and the nominal parameter vector θ^*

$$\theta^{*T} := (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in \mathbb{R}^{n+m} \tag{6.1.10}$$

we may rewrite (6.1.8) as

$$\hat{z}_p = \theta^{*T} \begin{bmatrix} \hat{w}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{w}_{n+m} \end{bmatrix} \tag{6.1.11}$$

or, in the time domain

$$z_p(t) = \theta^{*T} w(t) \tag{6.1.12}$$

where

$$w^T(t) := (w_1(t), \ldots, w_{n+m}(t)) \in \mathbb{R}^{n+m}$$

Note the close resemblance between (6.1.12) and the plant parameterization of (2.2.14) in Chapter 2. It is easy to verify that in the instance in which no prior information about the plant is available (case c) above), equation (6.1.12) above is a reformulation of equation (2.2.14).

The purpose of the identifier is to produce a recursive estimate $\hat{\theta}(t)$ of the parameter vector θ^*. Since r and y_p are available, the signals $z_p(t)$, $w(t)$ are obtainable through stable filtering of r and y_p, up to some exponentially decaying terms (since $\hat{N}_i$ and $\hat{D}_j$ are stable, proper, rational functions). These decaying terms will be neglected for simplicity.

In analogy to the expression of the plant equation (6.1.12), we define the output of the identifier as

$$z_i(t) := \theta^T(t) w(t) \tag{6.1.13}$$

We also define the parameter error

$$\phi(t) := \theta(t) - \theta^* \tag{6.1.14}$$

and the identifier error

$$e_1(t) := z_i(t) - z_p(t) \tag{6.1.15}$$

so that, for the analysis

$$e_1(t) = \phi^T(t) w(t) \tag{6.1.16}$$

Equation (6.1.16) is now exactly the same as (2.3.2) of Chapter 2, so that all the update algorithms and properties of Chapter 2 can be used verbatim in this context. Thus, for example, the *gradient algorithm*

$$\dot{\theta} = -g e_1 w \qquad g > 0 \tag{6.1.17}$$

or the *least-squares algorithm*

$$\begin{aligned} \dot{\theta} &= -gPe_1 w \\ \dot{P} &= -gPww^T P \qquad g > 0 \end{aligned} \tag{6.1.18}$$

along with covariance resetting, are appropriate parameter update laws. As in Chapter 2, the parameter error will converge (exponentially) to zero for the gradient or the least-squares algorithm with resetting if the vector w is persistently exciting, i.e. if there exist $\alpha_1, \alpha_2, \delta > 0$, such that

$$\alpha_2 I \geq \int_{t_0}^{t_0+\delta} w(\tau) w^T(\tau) d\tau \geq \alpha_1 I \qquad \text{for all } t_0 \geq 0 \tag{6.1.19}$$

Frequency Domain Conditions for Parameter Convergence

The techniques of Chapter 2 can be used to give frequency domain conditions on $r(t)$ to guarantee (6.1.19). To guarantee the upper bound, we simply assume:

(A1) Boundedness of the Regressor

The plant $\hat{P}(s)$ is stable, and the reference signal r is piecewise continuous and bounded.

As usual, the nontrivial condition comes from the lower bound in (6.1.19). To relate this condition on w to the frequency content of r, we will need a new *identifiability* condition on the transfer function from r to w, which we will denote

$$\hat{H}_{wr}^{T} := (\hat{N}_1(s), \ldots, \hat{N}_m(s), \hat{D}_1(s)\hat{P}(s), \ldots, \hat{D}_n(s)\hat{P}(s)) \qquad (6.1.20)$$

(A2) Identifiability

The system is assumed to be *identifiable*, meaning that for every choice of $n+m$ distinct frequencies $\omega_1, \ldots, \omega_{n+m}$, the vectors $\hat{H}_{wr}(j\omega_i) \in \mathbb{C}^{n+m}$ $(i = 1, \ldots, n+m)$ are linearly independent.

Proposition 6.1.1

Under assumptions (A1) and (A2), w is persistently exciting *if and only if* r is sufficiently rich of order $n+m$.

Proof of Proposition 6.1.1 similar to the proof of theorem 2.7.2.

Comments

a) From (6.1.20) and (6.1.11), it follows that if an input with $(n+m)$ spectral lines is applied to the system, we would have

$$(\hat{z}_p(j\omega_1), \ldots, \hat{z}_p(j\omega_{n+m})) = \theta^{*T}(\hat{H}_{wr}(j\omega_1), \ldots, \hat{H}_{wr}(j\omega_{n+m}))$$
$$\operatorname{diag}(\hat{r}(j\omega_1), \ldots, \hat{r}(j\omega_{n+m})) \qquad (6.1.21)$$

The identifiability condition implies that (6.1.21) has a unique solution for θ^*, while proposition 6.1.1 shows that the identifier parameter will converge to this value.

b) It is difficult to give a more concrete characterization of identifiability, since the components of $\hat{H}_{wr}(s)$ are proper stable rational functions of different orders. An exception is the case of example c) and discussed in Chapter 2. In that case, it was shown that the identifiability condition holds if the numerator and denominator of the plant transfer function are coprime polynomials.

6.1.2 Effect of Unmodeled Dynamics

The foregoing set up used transfer functions of the form (6.1.1) with $\hat{N}_i$, and $\hat{D}_j$ known exactly. In practice, the $\hat{N}_i$ and $\hat{D}_j$ are only known approximately. In particular, the transfer functions used to approximate the $\hat{N}_i$ and $\hat{D}_j$ will generally be low order, proper stable transfer functions (neglecting high-frequency dynamics and replacing near pole-zero cancellations by exact pole-zero cancellations). Thus, the identifier model of the plant is of the form

$$\hat{P}_a(s) = \frac{\hat{N}_{a0}(s) + \sum_{i=1}^{m} \alpha_i \hat{N}_{ai}(s)}{\hat{D}_{a0}(s) - \sum_{j=1}^{n} \beta_j \hat{D}_{aj}(s)} \qquad (6.1.22)$$

where $\hat{P}_a(s)$ is a proper stable transfer function, and $\hat{N}_{ao}$, $\hat{N}_{ai}$, $\hat{D}_{ao}$, $\hat{D}_{aj}$ are approximations of the actual transfer functions $\hat{N}_0$, $\hat{N}_i$, $\hat{D}_0$, $\hat{D}_j$. We will assume that

$$|\Delta\hat{N}_i(j\omega)| := |(\hat{N}_{ai} - \hat{N}_i)(j\omega)| < \epsilon \qquad (6.1.23)$$
$$\text{for all } \omega, \quad i = 0, \ldots, m$$

$$|\Delta\hat{D}_j(j\omega)| := |(\hat{D}_{aj} - \hat{D}_j)(j\omega)| < \epsilon \qquad (6.1.24)$$
$$\text{for all } \omega, \quad j = 0, \ldots, n$$

The identifier uses the form (6.1.22), while the true plant $\hat{P}(s)$ is accurately described by (6.1.1). Consequently, the signals of (6.1.9) are now replaced by

$$\hat{z}_{ap} := \hat{D}_{ao}\hat{y}_p - \hat{N}_{ao}\hat{r}$$
$$\hat{w}_{ai} := \hat{N}_{ai}\hat{r} \qquad i = 1, \ldots, m$$
$$\hat{w}_{am+j} := \hat{D}_{aj}\hat{y}_p \qquad j = 1, \ldots, n \qquad (6.1.25)$$

It is important to note that (6.1.12) is still valid, since the signals w, z_p pertain to the true plant. The identifier, however, uses the signals $w_{ai}(t)$ so that the identifier output

$$\bar{z}_i(t) := \theta^T(t)\, w_a(t) \qquad (6.1.26)$$

where $\theta(t)$ is the parameter estimate at time t. The parameter update laws of (6.1.17), (6.1.18) are modified by replacing w by w_a, while

$$e_1(t) = z_i(t) - z_p(t)$$

$$= \theta^T(t)\, w_a(t) - \theta^{*^T} w(t)$$

$$= \phi^T(t)\, w_a(t) + \theta^{*^T}(w_a(t) - w(t)) \tag{6.1.27}$$

Define

$$\Delta w(t) = w_a(t) - w(t)$$

Consequently, the gradient algorithm is described by

$$\dot\phi = \dot\theta = -g e_1 w_a$$

$$= -g w_a w_a^T \phi - g\theta^{*^T} \Delta w\, w_a \tag{6.1.28}$$

and the least-squares algorithm by

$$\dot\phi = \dot\theta = -gPe_1 w_a$$

$$= -gPw_a w_a^T \phi - gP\theta^{*^T} \Delta w\, w_a$$

$$\dot P = -gPw_a w_a^T P \tag{6.1.29}$$

Equations (6.1.28), (6.1.29) have a similar form as without unmodeled dynamics with the exception of the extra forcing terms

$$-g\theta^{*^T} \Delta w\, w_a \qquad \text{in (6.1.28)} \tag{6.1.30}$$

and

$$-gP\theta^{*^T} \Delta w\, w_a \qquad \text{in (6.1.29)} \tag{6.1.31}$$

Note that Δw_p is bounded, since it is the difference between the outputs of two proper transfer functions with a bounded reference input r. Consequently, the terms in (6.1.30) and (6.1.31) are bounded. Thus, if the systems of (6.1.28) and (6.1.29) are exponentially stable in the absence of the driving terms, the robustness results of Chapter 5 (specifically theorem 5.3.1) can be used to guarantee the convergence of the parameter error to a ball around the origin. It is further readily obvious from the estimates in the statement of the theorem that the size of the ball goes to zero as Δw shrinks (or, equivalently, the inaccuracy of modeling decreases).

It therefore remains to give conditions under which the undriven systems of (6.1.28), (6.1.29) with resetting are exponentially stable. It is easy to see that this is guaranteed if w_a is persistently exciting, i.e., condition (6.1.19) holds with w replaced by w_a. It is plausible that if ϵ (the extent of mismodeling of the $\hat N_i$, $\hat D_j$) is small enough and w is persistently exciting, then w_a is also persistently exciting. This is established in the following two lemmas.

Lemma 6.1.2 Persistency of Excitation under Perturbation

If the signal $w(t) \in \mathbb{R}^{n+m}$ is persistently exciting, i.e. there exist $\alpha_1, \alpha_2, \delta > 0$ such that

$$\alpha_2 I \geq \int_{t_0}^{t_0+\delta} w(\tau)\, w^T(\tau)\, d\tau \geq \alpha_1 I \quad \text{for all } t_0 \geq 0 \tag{6.1.32}$$

and the signal $\Delta w(t) \in \mathbb{R}^{n+m}$ satisfies

$$\sup_t\ |\Delta w(t)| < \left[\frac{\alpha_1}{\delta}\right]^{\frac{1}{2}} \tag{6.1.33}$$

Then $w + \Delta w$ is also persistently exciting.

Proof of Lemma 6.1.2

$w + \Delta w$ is persistently exciting if there exist $\alpha'_1, \alpha'_2, \delta' > 0$, such that for all $x \in \mathbb{R}^{n+m}$ of unit norm,

$$\alpha'_2 \geq \int_{t_0}^{t_0+\delta'} \left[x^T(w(\tau) + \Delta w(\tau))\right]^2 d\tau \geq \alpha'_1 \tag{6.1.34}$$

Let $\delta' = \delta$. The upper bound of the integral in (6.1.34) is automatically verified, simply because Δw is bounded and w satisfied a similar inequality. For the lower bound, we use the triangle inequality to get

$$\left[\int_{t_0}^{t_0+\delta} \left[x^T(w(\tau) + \Delta w(\tau))\right]^2 d\tau\right]^{\frac{1}{2}}$$

$$\geq \left[\int_{t_0}^{t_0+\delta} (x^T w(\tau))^2 d\tau\right]^{\frac{1}{2}} - \left[\int_{t_0}^{t_0+\delta} (x^T \Delta w(\tau))^2 d\tau\right]^{\frac{1}{2}}$$

$$\geq \alpha_1^{1/2} - \delta^{1/2} \sup_\tau\ |\Delta w(\tau)| \tag{6.1.35}$$

The conclusion now follows readily from (6.1.33). □

Thus, we see that w_a is guaranteed to be persistently exciting, when w persistently exciting and Δw is sufficiently small. The claim that Δw is small, when ϵ in (6.1.23), (6.1.24) is small enough, follows from the next lemma.

Lemma 6.1.3

If $\hat{g}(s)$ is a proper, stable, nth order rational function with corresponding impulse response $g(t)$

Then the L_1 norm of $g(t)$ can be bounded by

$$\| g \|_1 = \int_0^\infty | g(\tau) | \, d\tau \leq 2n \sup_\omega | \hat{g}(j\omega) | \tag{6.1.36}$$

Proof of Lemma 6.1.3 see Doyle [1984].

From lemma 6.1.3, and the definition of Δw, it is easy to verify that

$$| \Delta w(t) | \leq 2 \epsilon N \sup_\tau | r(\tau) | \tag{6.1.37}$$

where N is the maximum order of the ΔN_i, ΔD_j. Thus, $\Delta w(t)$ is small enough for ϵ small enough, and the persistency of excitation of w guarantees that of w_a. Using the estimate (6.1.37) and applying theorem 5.3.1, we see that the parameter error will converge to a ball of radius of order ϵ.

6.2 GLOBAL STABILITY OF INDIRECT ADAPTIVE CONTROL SCHEMES

The indirect approach is a popular technique of adaptive control. First, a non-adaptive controller is designed parametrically, that is, the controller parameters are written as functions of the plant parameters. Then, the scheme is made adaptive by replacing the plant parameters in the design calculation by their estimates at time t obtained from an on-line identifier. A reason for the popularity of the indirect approach is the considerable flexibility in the choice of both the controller and the identifier. Global stability of indirect schemes was shown by several authors in the discrete time case (Goodwin & Sin [1984], Anderson & Johnstone [1985], and others). In a continuous time context, Elliott, Cristi & Das [1985] used random parameter update times for proving convergence, and Kreisselmeier [1985, 1986] assumed that the parameters lie in a convex set in which no unstable pole-zero cancellations occur.

In Section 3.3.3, we considered the specific case of a model reference *indirect* adaptive control algorithm. We also indicated how it could be replaced by a pole placement algorithm for nonminimum phase systems in Section 3.3.5. In this section, we consider more general controller designs, following the approach of Bai & Sastry [1987]. We discuss a general, indirect adaptive control scheme for a SISO continuous time system using an identifier in conjunction with an *arbitrary* stabilizing controller. We will show that when the reference input is sufficiently rich, the input to the identifier is also sufficiently rich to cause parameter convergence in the identifier. The controller is updated only when adequate information has been obtained for a "meaningful" update. Thus, roughly speaking, the adaptive system consists of a fast parameter identification loop and a slow controller update loop. We will not need any conditions calling for the parameters to lie in a convex set or calling for lack of unstable pole-zero cancellations in the identifier. However, we will need sufficient richness conditions on the input that we were not assumed previously to establish global stability.

For application of the results, we will specialize our scheme to a pole placement adaptive controller, as well as to a factorization based adaptive stabilizer (of a kind that has attracted a great deal of interest in the literature on non-adaptive, robust control—see, for example, Vidyasagar [1985]).

6.2.1 Indirect Adaptive Control Scheme

The basic structure of an indirect adaptive controller is shown in Figure 6.4.

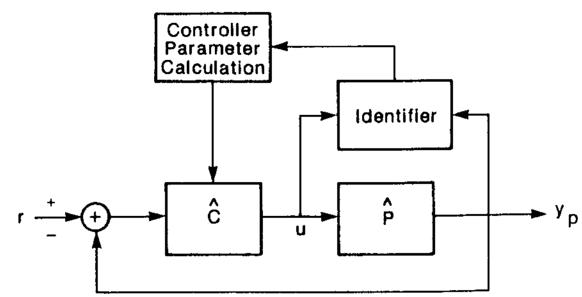

Figure 6.4 Basic Structure of an Indirect Adaptive Controller

The unknown, strictly proper plant is assumed to be described by

$$\hat{P}(s) = \frac{k_p \hat{n}_p(s)}{\hat{d}_p(s)} = \frac{\alpha_n s^{n-1} + \cdots + \alpha_1}{s^n + \beta_n s^{n-1} + \cdots + \beta_1} \tag{6.2.1}$$

with $\hat{n}_p(s)$, $\hat{d}_p(s)$ monic, coprime polynomials. The degree of $\hat{d}_p$ is n, and that of $\hat{n}_p$ is less than or equal to $n-1$ (consequently, some of the α_i's may be zero).

The compensator is a proper, m th order compensator of the form

$$\hat{C}(s) = \frac{\hat{n}_c(s)}{\hat{d}_c(s)} = \frac{a_{m+1}s^m + \cdots + a_1}{b_{m+1}s^m + \cdots + b_1} \tag{6.2.2}$$

The adaptive scheme proceeds as follows: the identifier obtains an estimate of the plant parameters. The compensator design (pole placement, model reference, etc.) is performed, assuming that the plant parameter estimates are the true parameter values (*certainty equivalence principle*). We will assume that there exists a unique compensator of the form (6.2.2) for every value of the plant parameter estimates. The hope is that, as $t \to \infty$, the identifier identifies the plant correctly, and therefore the compensator converges asymptotically to the desired one.

We first design an identifier as in Chapter 2. Define $w(t) \in \mathbb{R}^{2n}$ with Laplace transform

$$\hat{w}^T := \left[\frac{\hat{u}}{\hat{\lambda}}, \frac{s\,\hat{u}}{\hat{\lambda}}, \ldots, \frac{s^{n-1}\hat{u}}{\hat{\lambda}}, \frac{\hat{y}_p}{\hat{\lambda}}, \ldots, \frac{s^{n-1}\hat{y}_p}{\hat{\lambda}} \right] \tag{6.2.3}$$

with $\hat{\lambda}(s)$, a monic Hurwitz polynomial of the form $s^n + \lambda_n s^{n-1} + \cdots + \lambda_1$. Then

$$y_p(t) = \theta^{*^T} w(t) \tag{6.2.4}$$

where

$$\theta^{*^T} = \left[\alpha_1, \ldots, \alpha_n, \lambda_1 - \beta_1, \ldots, \lambda_n - \beta_n \right] \tag{6.2.5}$$

The identifier output is

$$y_i(t) = \theta^T(t)\, w(t) \tag{6.2.6}$$

where $\theta(t)$ is the estimate of θ^* at time t. If $\phi(t)$ is the parameter error $\phi(t) = \theta(t) - \theta^*$, then the identifier error $e_1(t) = y_i(t) - y_p(t)$ has the form

$$e_1(t) = \phi^T(t)\, w(t) \tag{6.2.7}$$

For the identification algorithm, we will use the least-squares with resetting

$$\dot{\phi}(t) = -P(t)\, w(t)\, e_1(t) \tag{6.2.8}$$

$$\dot{P}(t) = -P(t)\, w(t)\, w^T(t) P(t) \qquad t \neq t_i \tag{6.2.9}$$

with $P(t_i^+) = k_0 I > 0$, where t_i is the sequence of resetting times, *to be specified hereafter*. It is shown in the Appendix (lemma A6.2.1) that the parameter error $\phi(t)$ is bounded, even if $y_p(t)$ may not be. Further,

$\phi(t) \to 0$ asymptotically, if $w(t)$ is persistently exciting, i.e., if there exist $\alpha_1, \alpha_2, \delta > 0$, such that

$$\alpha_2 I \geq \int_{t_0}^{t_0+\delta} w(\tau)\, w^T(\tau)\, d\tau \geq \alpha_1 I \qquad \text{for all } t_0 \geq 0 \tag{6.2.10}$$

Note however that the upper bound in (6.2.10) is not needed for the specific algorithm chosen here. Further, it has been shown in Chapter 2 that under the condition that $\hat{n}_p$, $\hat{d}_p$ are coprime polynomials, w satisfies the lower bound condition (6.2.10) if u is sufficiently rich, i.e., if the support of the spectrum of u has at least $2n$ points (assuming that $u(t)$ is stationary).

The design of the compensator is based on the plant parameter estimates $\theta(t)$. It would appear intuitive that if $\theta(t) \to \theta^*$ as $t \to \infty$, then the time-varying compensator will converge to the nominal compensator and the closed-loop system will be asymptotically stable. Therefore, the system of Figure 6.4 can be understood as a time-varying linear system which is asymptotically time-invariant and stable. The following lemma guarantees the asymptotic stability of the linear time varying system.

Lemma 6.2.1
Consider the time-varying system

$$\dot{x} = (A + \Delta A(t))\, x \tag{6.2.11}$$

where A is a constant matrix with eigenvalues in the open LHP and $\| \Delta A(t) \|$ is a bounded function of t converging to zero as $t \to \infty$.

Then (6.2.11) is asymptotically stable, i.e. there exist $m, \alpha > 0$ such that the state transition matrix $\Phi(t, t_0)$ of $A + \Delta A(t)$ satisfies

$$\| \Phi(t, t_0) \| \leq m e^{-\alpha(t - t_0)} \qquad \text{for all } t > t_0$$

Update Sequence for the Controller
Although the update law (6.2.8), (6.2.9) may be shown to be asymptotically stable when w satisfies (6.2.10) (in fact only the lower bound), it is of practical importance to limit the update of the controller to instants when sufficient new information has been obtained. This is measured through the "information matrix"

$$\int_{t_i}^{t_i+\delta} w(\tau)\, w^T(\tau)\, d\tau$$

Thus, given $\gamma > 0$, we choose the update sequence t_i by $t_0 = 0$, and $t_{i+1} = t_i + \delta_i$, where δ_i satisfies

$$\delta_i := \operatorname*{argmin}_{\Delta} \int_{t_i}^{t_i + \Delta} w\, w^T d\tau \;\geq\; \gamma I \tag{6.2.12}$$

Then, the compensator parameters are held constant between t_i and t_{i+1}. We will assume that the compensator parameters are continuous functions of θ. We may now relate the richness of the reference signal $r(t)$ in Figure 6.4 to the convergence of the identifier.

Lemma 6.2.2 Convergence of the Identifier

Consider the system of Figure 6.4 with the least-squares update law (6.2.8), (6.2.9) and resetting times identical to the controller update times given in (6.2.12).

If the input r is stationary and its spectral support contains at least $3n + m$ points,

Then the identifier parameter error converges to zero exponentially as $t \to \infty$. More precisely, there exists $0 < \xi < 1$ such that

$$|\,\phi(t_i)\,| \;\leq\; \xi^i |\,\phi(0)\,| \tag{6.2.13}$$

and $\delta_i := t_{i+1} - t_i$ is a bounded sequence.

Proof of Lemma 6.2.2

The proof uses lemmas which are collected in the Appendix. In particular, lemma A6.2.1 shows the conclusion (6.2.13) if the sequence δ_i is bounded. Thus, we will establish this fact. We proceed by contradiction. If δ_i is an unbounded sequence, then one of two following possibilities occurs

(a) There exists an $i < \infty$ such that $\delta_i = \infty$ or

(b) $\delta_i \to \infty$ as $i \to \infty$.

Consider the scenario (a) first. If this happens the system becomes time invariant after t_i, since the controller is not updated. Consequently, one can find the transfer function from r to u to be

$$\hat{H}_{ur} = \frac{k_p \hat{n}_c(t_i)\hat{n}_p}{k_p \hat{n}_p \hat{n}_c(t_i) + \hat{d}_p \hat{d}_c(t_i)} := \frac{\hat{n}}{\hat{d}} \tag{6.2.14}$$

where $\hat{d}_c(t_i)$ and $\hat{n}_c(t_i)$ are the denominator and numerator of the controller at time t_i. Using (6.2.14), we may write the transfer function

from r to w to be

$$\hat{H}_{wr}(s) = \frac{\hat{n}}{\hat{\lambda}\hat{d}_p\hat{d}} \left[\hat{d}_p, \ldots, s^{n-1}\hat{d}_p, k_p\hat{n}_p, \ldots, k_p\hat{n}_p s^{n-1}\right]^T \tag{6.2.15}$$

Since the degree of $\hat{n}$ is $(n+m)$, no more than $(n+m)$ of the spectral lines of the input can correspond to zeros of the numerator polynomial. Even in this (worst-case) situation, we see from the arguments of Chapter 2 that w is persistently exciting. This fact, however, contradicts the assumption that $\delta_i = \infty$.

Now, consider scenario (b). First, notice that when the plant parameters are known, the closed loop system is time-invariant and stable so that we may write the equation relating $r(t)$ to the signal $w_m(t)$ ($w_m(t)$ corresponds to $w(t)$ in the case when $\phi(t) = 0$, as in Chapter 3)

$$\dot{z}_m = A z_m + b r$$
$$w_m = C z_m$$

where A, C are constant matrices, b is a constant vector and A is stable. For the adaptive control situation, $\phi(t) \neq 0$, but we may still write the following equation relating $r(t)$ to $w(t)$

$$\dot{z} = (A + \Delta A(t))z + (b + \Delta b(t))r$$
$$w = (C + \Delta C(t))z$$

where $\Delta A(t), \Delta C(t), \Delta b(t)$ are continuous functions of $\phi(t)$ and $\Delta A(t), \Delta C(t)$ and $\Delta b(t) \to 0$ as $\phi(t) \to 0$. If scenario (b) happens, we still have that $\phi(t) \to 0$ as $t \to \infty$ from lemma A6.2.1. Further, from lemma A6.2.2 it follows that $w_m(t)$ approaches $w(t)$ for t large enough. Then the persistency of excitation of $w(t)$ follows as a consequence of the result of lemma A6.2.2 and the fact that $w_m(t)$ is persistently exciting. This, however, contradicts the hypothesis that $\delta_i \to \infty$ as $i \to \infty$. ∞

□

Theorem 6.2.3 Asymptotic Stability of the Indirect Adaptive System

Consider the system of Figure 6.4, with the plant and compensator as in lemma 6.2.2.

If the inpuf r is stationary and its spectral support contains at least $3n + m$ points

Then the adaptive system is asymptotically stable.

Proof of Theorem 6.2.3 follows readily from lemmas 6.2.1 and 6.2.2.

6.2.2 Indirect Adaptive Pole Placement

Pole placement is easily described in the context of Figure 6.4. The compensator $\hat{C}$ is chosen so that the closed-loop poles lie at the zeros of a given characteristic polynomial $\hat{d}_{\rm cl}(s)$, typically of degree $(2n-1)$. In other words, $\hat{n}_c$, $\hat{d}_c$ need to be found to satisfy

$$k_p\hat{n}_c\hat{n}_p + \hat{d}_c\hat{d}_p = \hat{d}_{\rm cl} \tag{6.2.16}$$

When $\hat{n}_p$, $\hat{d}_p$ are coprime, equation (6.2.16) may be solved (see lemma A6.2.3 in the Appendix) for any arbitrary $\hat{d}_{\rm cl}$ of degree $(2n-1)$, with $\hat{n}_c$, $\hat{d}_c$ each of order $n-1$.

When the plant is unknown, the "adaptive" pole placement scheme is mechanized by using the estimates $\hat{n}_p(t_i)$, $\hat{d}_p(t_i)$ of the numerator and denominator polynomials. Using lemma A6.2.3 again, it is easy to verify that if $k_p(t_i)\,\hat{n}_p(t_i)$, $\hat{d}_p(t_i)$ are coprime, there exist unique $\hat{n}_c(t_i)$ and $\hat{d}_c(t_i)$ of order $n-1$ such that

$$k_p(t_i)\,\hat{n}_c(t_i)\hat{n}_p(t_i) + \hat{d}_c(t_i)\hat{d}_p(t_i) = \hat{d}_{\rm cl} \tag{6.2.17}$$

The estimates for $k_p(t_i)\,\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ follow from the plant parameter estimates $\theta(t)$ of the identifier. In analogy to the plant parameter vector θ, we have the parameter vector of the compensator (cf. (6.2.2) with $m = n-1$)

$$\theta_c(t) = \Big[b_0(t), \ldots, b_n(t), a_0(t), \ldots, a_n(t)\Big] \tag{6.2.18}$$

As usual, θ_c^* has the interpretation of being the nominal compensator parameter vector. Further, to guarantee that $k_p(t_i)\,\hat{n}_p(t_i)$ and $\hat{d}_p(t_i)$ are coprime at the instants t_i, we need to modify the definitions of the update times. Let

$$t_{i+1} = t_i + \delta_i \tag{6.2.19}$$

where δ_i is the smallest real number satisfying

$$\int_{t_i}^{t_i+\delta_i} ww^T dt \geq \gamma I \tag{6.2.20}$$

$$k_p(t_i)\,\hat{n}_p(t_i+\delta_i) \text{ and } \hat{d}_p(t_i+\delta_i) \text{ are coprime} \tag{6.2.21}$$

More precisely (6.2.21) is satisfied by guaranteeing that the smallest singular value of the matrix in lemma A6.2.3 of the Appendix—measuring the coprimeness of $k_p(t_i+\delta_i)\,\hat{n}_p(t_i+\delta_i)$, $\hat{d}_p(t_i+\delta_i)$—exceeds a number $\sigma > 0$. □

Theorem 6.2.4 Asymptotic Stability of the Adaptive Pole Placement Scheme

Consider the adaptive pole placement scheme, with the least squares identifier of (6.2.8)–(6.2.9) and the update sequence t_i defined by (6.2.19)–(6.2.21).

If the input $r(t)$ is stationary and its spectral support contains at least $4n-1$ points

Then all signals in the loop are bounded and the characteristic polynomial of the closed loop system tends to $\hat{d}_{\rm cl}(s)$. Moreover, $|\,\theta_c(t)-\theta_c^*\,| \to 0$ exponentially.

Proof of Theorem 6.2.4

The first half of the theorem follows from lemmas 6.2.1, 6.2.2—a slight modification of the arguments of lemma 6.2.2 is needed to account for the new condition (6.2.21) in the update time, but this is easy because of the convergence of the identifier and the coprimeness of the *true* $\hat{n}_p$, $\hat{d}_p$. For the second half, we see from lemma A6.2.3 in the Appendix that

$$A(\theta(t_i))\,\theta_c(t_i) = d \tag{6.2.22}$$

with d the vector of coefficients of the polynomial $\hat{d}_{\rm cl}$. It follows from lemma A6.2.3 in the Appendix that there is an $m > 0$ such that

$$\|A(\theta(t_i)) - A(\theta^*)\| \leq m|\,\theta(t_i)-\theta^*|$$

Further, subtracting (6.2.22) from

$$A(\theta^*)\theta_c^* = d \tag{6.2.23}$$

we see that

$$\Big[A(\theta(t_i)) - A(\theta^*)\Big]\,\theta_c(t_i) = -A(\theta^*)(\theta_c(t_i)-\theta_c^*)$$

Thus

$$|\,\theta_c(t_i)-\theta_c^*\rfloor \leq m\|A^{-1}(\theta^*)\|\;|\,\theta(t_i)-\theta^*\,|\;|\,\theta_c(t_i)\,| \tag{6.2.24}$$

Since $|\,\theta_c(t_i)\,|$ is bounded (by (6.2.21)), we get that

$$|\,\theta_c(t_i)-\theta_c^*| \leq m_1|\,\theta(t_i)-\theta^*| \tag{6.2.25}$$

for some m_1. Since $\theta(t_i)$ converges to θ^* exponentially, so does $\theta_c(t_i)$ to

θ_c^*. □

6.2.3 Indirect Adaptive Stabilization—The Factorization Approach

The Factorization Approach to Controller Design

We will briefly review the factorization approach to controller design (the non-adaptive version). Consider the controller structure of Figure 6.5.

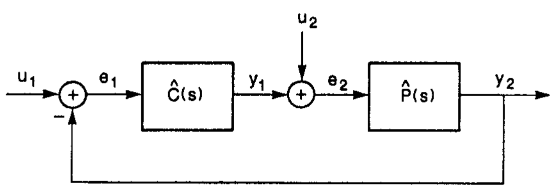

Figure 6.5 Standard One Degree of Freedom Controller

The plant $\hat{P}$ is as defined in (6.2.1) and the compensator $\hat{C}$ as in (6.2.2). The transfer function relating e_1, e_2 to u_1, u_2 is given by

$$\hat{H}_{eu} = \frac{1}{1+\hat{P}\hat{C}} \begin{bmatrix} 1 & -\hat{P} \\ \hat{C} & 1 \end{bmatrix} \tag{6.2.26}$$

The system of Figure 6.5 is BIBO stable if and only if each of the four elements of (6.2.26) is stable, i.e., belongs to $\boldsymbol{R}$, the ring of proper, stable, rational functions. The ring $\boldsymbol{R}$ is a more convenient ring than is the ring of polynomials for several reasons (see for example, Vidyasagar [1985])—including the study of the robustness properties of the closed loop systems. We assumed that $\hat{P}$ and $\hat{C}$ are factored coprimely in $\boldsymbol{R}$ (not uniquely!) as

$$\hat{P}(s) = \hat{N}_p(s)\hat{D}_p^{-1}(s)$$

$$\hat{C}(s) = \hat{D}_c^{-1}(s)\hat{N}_c(s) \tag{6.2.27}$$

From Vidyasagar [1985], it follows that the system of Figure 6.5 is BIBO stable if and only if $(\hat{N}_p\hat{N}_c + \hat{D}_p\hat{D}_c)^{-1}$ belongs to $\boldsymbol{R}$, or, equivalently, $\hat{N}_p\hat{N}_c + \hat{D}_p\hat{D}_c$ is a unimodular element of $\boldsymbol{R}$. Without loss of generality, we can state that a compensator stabilizes the system of Figure 6.5 if and only if

$$\hat{N}_p\hat{N}_c + \hat{D}_p\hat{D}_c = 1 \tag{6.2.28}$$

Equation (6.2.28) parameterizes all stabilizing compensators. We will be interested in a parameterization of all solutions of (6.2.28) in terms of the coefficients of $\hat{N}_p$, $\hat{D}_p$. For this purpose, let (A_p, b_p, c_p^T) be a minimal realization of $\hat{P}(s)$. If $f \in \mathbb{R}^n$ and $l \in \mathbb{R}^n$ are chosen to stabilize $A_{pf} := A_p - b_p f^T$ and $A_{pl} := A_p - l c_p^T$ (such a choice is possible by the minimality of the realization of A_p, b_p, c_p^T), then, it may be shown (see Vidyasagar [1985]) that a right coprime fraction of $\hat{P}$ is given by

$$\hat{N}_p = c_p^T(sI - A_{pf})^{-1}b_p \tag{6.2.29}$$

$$\hat{D}_p = 1 - c_p^T(sI - A_{pl})^{-1}b_p \tag{6.2.30}$$

and further that all solutions of (6.2.28) may be written as

$$\hat{D}_c = 1 + c_p^T(sI - A_{pf})^{-1}l - \hat{Q}(s)\,c_p^T(sI - A_{pf})^{-1}b_p \tag{6.2.31}$$

$$\hat{N}_c = f^T(sI - A_{pf})^{-1}l + \hat{Q}(s)\left[1 - f^T(sI - A_{pf})^{-1}b_p\right] \tag{6.2.32}$$

with $\hat{Q}(s) \in \boldsymbol{R}$ an arbitrary element chosen to meet other performance criteria (such as minimization of the disturbance to output map, obtaining the desired closed loop transfer function, optimal desensitization to unmodeled dynamics, etc.).

The optimal choice of $\hat{Q}(s)$ depends on the plant parameters. However, such a choice of $\hat{Q}(s)$ may not be unique or depend continuously on plant parameters. The optimal choice of $\hat{Q}(s)$ is the topic of the so-called H^∞ *optimal control* systems design methodology. In this chapter, we will not concern ourselves with anything more than stabilization, and use a fixed $\hat{Q}(s)$ rather than one whose calculation depends on the current estimate of plant parameters. For simplicity we will, in fact, fix $\hat{Q}(s) = 0$ in the adaptive stabilization which follows.

Adaptive Stabilization Using the Factorization Approach

The objective is to design the compensator $\hat{C}(s)$ adaptively based on the estimate θ of the plant parameters so that the closed loop system is asymptotically stable with all signals bounded. Set $u_2(t) \equiv 0$.

The identifier and compensator update sequence $\{t_i\}$ are specified in (6.2.19)–(6.2.21). The only difficulty in mechanizing the n th order compensator of (6.2.31) and (6.2.32) (with $\hat{Q}(s) = 0$) lies in calculating

$f(t_i), l(t_i) \in \mathbb{R}^n$ to stabilize $A_p(t_i)$, the estimate of A_p based on the current plant parameter estimate. To that effect, we consider the controllable form and observable form realizations of the plant estimate

$$A_p(t_i) = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & 1 \\ -\beta_1(t_i) & -\beta_2(t_i) & & -\beta_n(t_i) \end{bmatrix} \qquad b_p = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{6.2.33}$$

$$c_p^T(t_i) = [\alpha_1(t_i)\,\alpha_2(t_i)\,\cdots\,\alpha_n(t_i)]$$

and

$$\tilde{A}_p(t_i) = \begin{bmatrix} 0 & 0 & 0 & -\beta_1(t_i) \\ 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 1 & -\beta_n(t_i) \end{bmatrix} \qquad \tilde{b}_p(t_i) = \begin{bmatrix} \alpha_1(t_i) \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n(t_i) \end{bmatrix} \tag{6.2.34}$$

$$\tilde{c}_p^T(t_i) = [0\ \cdots\ 0\ 1]$$

Let the transformation matrix $M(t_i)$ relate the realization (6.2.33) to (6.2.34)

$$\tilde{A}_p(t_i) = M(t_i)\,A_p M(t_i)^{-1}$$

$$\tilde{b}_p(t_i) = M(t_i)\,b_p(t_i)$$

$$\tilde{c}_p^T(t_i) = c_p^T(t_i)M(t_i)^{-1}$$

Note that $M(t_i)$ is the only calculation that needs to be performed. Now $f(t_i)$ and $l(t_i)$ are easily read off. Indeed, consider any stable polynomial $s^n + p_n s^{n-1} + \cdots + p_1$. Then it is easy to see that

$$f^T(t_i) = [-p_1 + \beta_1(t_i), \ldots, -p_n + \beta_n(t_i)]^T$$

and

$$l(t_i) = M^{-1}(t_i)[-p_1 + \beta_1(t_i), \ldots, -p_n + \beta_n(t_i)]^T \tag{6.2.35}$$

Therefore $A_p(t_i) - b_p f^T(t_i)$ and $A_p(t_i) - l(t_i)c_p^T(t_i)$ have their eigenvalues at the zeros of $s^n + p_n s^{n-1} + \cdots + p_1$. The compensator of (6.2.31), (6.2.32) with $\hat{Q}(s) = 0$ can be made adaptive by choosing $\hat{C}(t_i) = \hat{D}_c^{-1}(t_i)\,\hat{N}_c(t_i)$ with

$$\hat{N}_c(t_i) = f^T(t_i)(sI - A_{pf}(t_i))^{-1} l(t_i) \tag{6.2.36}$$

$$\hat{D}_c(t_i) = 1 + c_p^T(t_i)(sI - A_{pf}(t_i))^{-1} l(t_i) \tag{6.2.37}$$

Then, as before we have the following theorem:

Theorem 6.2.5 Asymptotic Stability of the Adaptive Identifier Using the Factorization Approach

Consider the set up of Figure 6.5, with the least squares identifier of (6.2.8), (6.2.9), the update sequence $\{t_i\}$ of (6.2.19)–(6.2.21), and the compensator of (6.2.36), (6.2.37).

If the input is stationary and its spectral support is not concentrated on $k \le 4n$ points,

Then the adaptive system is asymptotically stable.

Proof of Theorem 6.2.5 follows as the proof of theorem 6.2.4.

6.3 MULTIVARIABLE ADAPTIVE CONTROL

6.3.1 Introduction

The extension of adaptive control algorithms for single-input single-output systems (SISO) to multi-input multi-output systems (MIMO) is far from trivial. Indeed, transfer function properties which are easily established for SISO systems are much more complex for MIMO systems. The issue of the parameterization of the adaptive controllers becomes a dominant problem. Several MIMO adaptive control algorithms were proposed based on the model reference approach (Singh & Narendra [1982], Elliott & Wolovich [1982], Goodwin & Long [1980], Johansson [1987]), the pole placement approach (Prager & Wellstead [1981], Elliot, Wolovich & Das [1984]), and quadratic optimization approaches (Borisson [1979], Koivo [1980]). See also the review/survey papers by Dugard & Dion [1985] and Elliott & Wolovich [1984].

The understanding of parameterization issues benefited significantly from progress in the theory of nonadaptive MIMO control theory. In Section 6.3.2, we briefly review some basic results. More details may be found in Kailath [1980], Callier & Desoer [1984], and Vidyasagar [1985]. These results will help us to establish the parameterization of MIMO adaptive controllers in Section 6.3.3. Once the parameterization is established, the design of adaptive schemes will follow in a more straightforward manner from SISO theory (Section 6.3.4).

As previously, we will concentrate our discussion on a model reference adaptive control scheme and follow an approach parallel to that of Chapter 3. Alternate adaptive control schemes may be found in the references just mentioned. We will say very little about the dynamic properties of MIMO adaptive control systems. Indeed, this topic is not well understood so far (even less than for SISO systems!).

6.3.2 Preliminaries

6.3.2.1 Factorization of Transfer Function Matrices

Right and Left Fractions

Consider a square transfer function matrix $\hat{P}(s)$, with p rows and p columns, whose elements are rational functions of s with real coefficients. The set of such matrices is denoted $\hat{P}(s) \in \mathbb{R}^{p\times p}(s)$. A rational transfer function is expressed as the ratio of two polynomials in s. Similarly, a rational transfer function matrix may be represented as the ratio of two polynomial matrices. The set of polynomial matrices of dimension $p \times p$ is denoted $\mathbb{R}^{p\times p}[s]$ (note the slight difference in notation). A pair of polynomial matrices $(\hat{N}_R, \hat{D}_R)$ is called a *right fraction (r.f.)* of $\hat{P}(s) \in \mathbb{R}^{p\times p}(s)$ if

- $\hat{N}_R, \hat{D}_R \in \mathbb{R}^{p\times p}[s]$
- $\hat{D}_R$ nonsingular, i.e., $\det(\hat{D}_R(s)) \neq 0$ for almost all s.
- $\hat{P} = \hat{N}_R \hat{D}_R^{-1}$

Similarly, a pair $(\hat{D}_L, \hat{N}_L)$ is called a *left fraction (l.f.)* of $\hat{P}(s) \in \mathbb{R}^{p\times p}(s)$ *if*

- $\hat{N}_L, \hat{D}_L \in \mathbb{R}^{p\times p}[s]$
- $\hat{D}_L$ nonsingular, i.e. $\det(\hat{D}_L(s)) \neq 0$ for almost all s
- $\hat{P} = \hat{D}_L^{-1} \hat{N}_L$

Given a right fraction $(\hat{N}_R, \hat{D}_R)$, another right fraction may be obtained by multiplying $\hat{N}_R$ and $\hat{D}_R$ on the right by a nonsingular polynomial matrix $\hat{R}(s) \in \mathbb{R}^{p\times p}[s]$, i.e.

$$\begin{array}{lll} \hat{N}_R, \hat{D}_R \ \text{r.f. of } \hat{P} & \Longrightarrow \ \hat{N}_{R_1} = \hat{N}_R \hat{R} & \text{r.f. of } \hat{P} \\ \hat{R} \ \text{nonsingular} & \qquad \hat{D}_{R_1} = \hat{D}_R \hat{R} & \end{array}$$

The matrix $\hat{R}$ is called a *common right divisor* of $\hat{N}_{R_1}$ and $\hat{D}_{R_1}$. It is called the *greatest common right divisor (gcrd)* of $\hat{N}_{R_1}$ and $\hat{D}_{R_1}$, if any other common right divisor of $\hat{N}_{R_1}$ and $\hat{D}_{R_1}$ is also a common right divisor of $\hat{R}$. In fact, "the" gcrd is not unique, but all gcrd's are equivalent in a sense to be defined hereafter.

Similar definitions follow for a left fraction $(\hat{D}_L, \hat{N}_L)$. Given $(\hat{D}_L, \hat{N}_L)$ a left fraction of $\hat{P}(s)$ and a nonsingular matrix $\hat{L}$, $(\hat{L}\hat{D}_L, \hat{L}\hat{N}_L)$ is also a left fraction of $\hat{P}(s)$. Greatest common left divisors are defined by making the appropriate transpositions.

Right and Left Coprime Fractions—Poles and Zeros

A polynomial matrix $\hat{D}_R(s)$ is called *unimodular* if its inverse is a polynomial matrix. A necessary and sufficient condition is that $\det \hat{D}_R(s)$ is a real number different from 0. Clearly, a unimodular matrix is nonsingular. Further, multiplying a polynomial matrix by a unimodular matrix does not affect its rank or the degree of its determinant.

Two matrices $\hat{R}_1$ and $\hat{R}_2$ are said to be *right equivalent* if there exists a unimodular matrix $\hat{R}$ such that $\hat{R}_2 = \hat{R}_1 R$. Given a pair $(\hat{N}_R, \hat{D}_R)$ and a gcrd $\hat{R}_1$, the matrix $\hat{R}_2 = \hat{R}_1 \hat{R}$ is also a gcrd if $\hat{R}$ is unimodular. In fact, it may be shown that all gcrd's are so related, that is, *all gcrd's are right equivalent.* Extracting the gcrd $\hat{R}$ of a right fraction $(\hat{N}_{R_1}, \hat{D}_{R_1})$ is pretty much like extracting common factors in a scalar transfer function. The new pair $(\hat{N}_R = \hat{N}_{R_1}\hat{R}^{-1}, \hat{D}_R = \hat{D}_{R_1}\hat{R}^{-1})$ is also a right fraction so that $\hat{P} = \hat{N}_{R_1} D_{R_1}^{-1} = \hat{N}_R \hat{D}_R^{-1}$. Note that the gcrd's of $(\hat{N}_R, \hat{D}_R)$ are unimodular. In general, two matrices $(\hat{N}_R, \hat{D}_R)$ are called *right coprime* if their gcrd is unimodular. It may be shown that for all $\hat{P}(s) \in \mathbb{R}^{p\times p}(s)$, there exists a *right coprime fraction* of $\hat{P}(s)$; that is,

- $\hat{N}_R, \hat{D}_R \in \mathbb{R}^{p\times p}[s]$ right coprime
- $\hat{D}_R$ nonsingular
- $\hat{P} = \hat{N}_R \hat{D}_R^{-1}$

In analogy to the SISO case, the poles and zeros of $\hat{P}$ are defined as

- p is a *pole* of $\hat{P}$ if $\det(\hat{D}_R(p)) = 0$
- z is a *zero* of $\hat{P}$ if $\det(\hat{N}_R(z)) = 0$

where $(\hat{N}_R, \hat{D}_R)$ is a right coprime fraction of $\hat{P}$. Similarly, n = order of the system $= \partial \det(\hat{D}_R(s))$. It may be shown that these definitions correspond to the similar definitions for a minimal state-space realization of a proper $\hat{P}(s)$.

Similar definitions are found for left coprime fractions. It follows that rcf and lcf of a plant $\hat{P}$ must satisfy $\det(\hat{D}_R(s)) = \det(\hat{D}_L(s))$ and $\det(\hat{N}_R(s)) = \det(\hat{N}_L(s))$, except for a constant real factor.

Properness, Column Reduced Matrices

We restrict our attention to proper and strictly proper transfer function matrices, defined as follows:

$$\hat{P}(s) \in \mathbb{R}_p^{p\times p}(s) \quad \text{if} \quad \lim_{s\to\infty} \hat{P}(s) \text{ exists}$$

$$\hat{P}(s) \in \mathbb{R}_{p,o}^{p\times p}(s) \quad \text{if} \quad \lim_{s\to\infty} \hat{P}(s) = 0$$

Consider a right fraction $(\hat{N}_R, \hat{D}_R)$, and define the *column degrees* as

$$\partial_{cj}(\hat{D}_R) = \max_i [\partial(\hat{D}_R)_{ij}]$$

The following fact is easily established:

$$\hat{P}(s) \in \mathbb{R}_p^{p\times p}(s) \implies \partial_{cj}(\hat{N}_R) \le \partial_{cj}(\hat{D}_R)$$

$$\hat{P}(s) \in \mathbb{R}_{p,o}^{p\times p}(s) \implies \partial_{cj}(\hat{N}_R) < \partial_{cj}(\hat{D}_R)$$

The converse is true if we introduce the concept of column reduced matrices. First define the *highest column degree coefficient matrix* D_{hc}

$$(D_{hc})_{ij} = \text{coefficient of the term of degree } \partial_{cj}(\hat{D}_R) \text{ in } (D_R)_{ij}$$

A matrix is called *column reduced* (also *column proper*) if D_{hc} is nonsingular. If $(\hat{N}_R, \hat{D}_R)$ is a right fraction of $\hat{P}(s)$ and $\hat{D}_R$ is column reduced, then

$$\partial_{cj}(\hat{N}_R) \le \partial_{cj}(\hat{D}_R) \iff \hat{P}(s) \in \mathbb{R}_p^{p\times p}(s)$$

$$\partial_{cj}(\hat{N}_R) < \partial_{cj}(\hat{D}_R) \iff \hat{P}(s) \in \mathbb{R}_{p,o}^{p\times p}(s)$$

If $(\hat{N}_R, \hat{D}_R)$ is a right *coprime* fraction of $\hat{P}(s)$, and $\hat{D}_R$ is column reduced, we call

$$\mu_j = \partial_{cj}(\hat{D}_R) := \textit{controllability indices of } \hat{P}$$

$$\mu = \max_j (\mu_j) := \textit{controllability index of } \hat{P}$$

It is a remarkable fact that $\{\mu_j\}$ is invariant (see Kailath [1980]). In other words, the controllability indices are the same (modulo permutations) for all r.c.f. $(\hat{N}_R, \hat{D}_R)$ with $\hat{D}_R$ column reduced. Note also that

$$n = \text{order of the system} = \sum_{j=1}^{p} \mu_j$$

It may also be shown that the definition of controllability indices correspond to the alternate definition for a minimal state-space realization.

Given a matrix $\hat{P}(s) \in \mathbb{R}_p^{p\times p}(s)$ or $\hat{P}(s) \in \mathbb{R}_{p,o}^{p\times p}(s)$, it is always possible to find a rcf $(\hat{N}_R, \hat{D}_R)$ such that $\hat{D}_R$ is column reduced. The procedure is somewhat lengthy and is discussed in Kailath [1980]. The first step is to obtain a right coprime fraction (using a Hermite row form decomposition), and the second step is to reduce the matrix to a column reduced form, multiplying further on the right by some appropriately chosen unimodular matrix.

Properness, Row Reduced Matrices

Similar facts and definitions hold for left fractions and are briefly summarized. The *highest row degree coefficient matrix* D_{hr} is defined as:

$$(D_{hr})_{ij} = \text{coefficient of the term of degree } \partial_{ri}(\hat{D}_L) \text{ in } (\hat{D}_L)_{ij}$$

$$\text{where} \quad \partial_{ri}(\hat{D}_L) = \max_j (\partial(\hat{D}_L)_{ij}) \text{ are the } \textit{row degrees} \text{ of } \hat{D}_L.$$

The matrix $\hat{D}_L$ is called *row reduced* if D_{hr} is nonsingular. If $(\hat{D}_L, \hat{N}_L)$ is a left fraction of $\hat{P}(s)$, and $\hat{D}_L$ is row reduced, then

$$\partial_{ri}(\hat{N}_L) \le \partial_{ri}(\hat{D}_L) \iff \hat{P}(s) \in \mathbb{R}_p^{p\times p}(s)$$

$$\partial_{ri}(\hat{N}_L) < \partial_{ri}(\hat{D}_L) \iff \hat{P}(s) \in \mathbb{R}_{p,o}^{p\times p}(s)$$

When $(\hat{D}_L, \hat{N}_L)$ is a left *coprime* fraction of $\hat{P}(s)$, with $\hat{D}_L$ row reduced, we define

$$\nu_i = \partial_{ri}(\hat{D}_L) := \textit{observability indices} \text{ of } \hat{P}$$

$$\nu = \max_i (\nu_i) := \textit{observability index} \text{ of } \hat{P}$$

The set of observability indices is invariant. They are different from the controllability indices, although related by

$$n = \text{system order} = \sum_{i=1}^{p} \nu_i = \sum_{j=1}^{p} \mu_j$$

Polynomial Matrix Division

Consider, first, scalar polynomials. Given two polynomials $\hat{n}$ and $\hat{d}$, the standard division algorithm provides $\hat{q}$ and $\hat{r}$ such that

$$\hat{n} = \hat{q}\,\hat{d} + r \qquad \partial\hat{r} < \partial\hat{d}$$

This procedure is equivalent to separating the strictly proper and not strictly proper parts of a transfer function:

$$\frac{\hat n}{\hat d} = \hat q + \frac{\hat r}{\hat d} \qquad \hat q \text{ polynomial}, \ \frac{\hat r}{\hat d} \text{ strictly proper}$$

From this observation, the following proposition is obtained in the multivariable case.

Proposition 6.3.1 Polynomial Matrix Division

Let $\hat N_R, \hat D_R, \hat N_L, \hat D_L \in \mathbb{R}^{p\times p}[s]$ with $\hat D_R, \hat D_L$ nonsingular. ~~Let $\hat D_R$ be column reduced and $\hat D_L$ be row reduced.~~

Then There exists $\hat Q_R, \hat R_R, \hat Q_L, \hat R_L \in \mathbb{R}^{p\times p}[s]$ such that

$$\hat N_R = \hat Q_R \hat D_R + \hat R_R \qquad \partial_{cj}(\hat R_R) < \partial_{cj}(\hat D_R)$$

$$\hat N_L = \hat D_L \hat Q_L + \hat R_L \qquad \partial_{ri}(\hat R_L) < \partial_{ri}(\hat D_L)$$

Proof of Proposition 6.3.1

The elements of $\hat N_R \hat D_R^{-1}$ are rational functions of s. Divide each numerator by its denominator and call the matrix of quotients $\hat Q_R$. Therefore $\hat N_R \hat D_R^{-1} = \hat Q_R + \hat S_R$ where $\hat S_R \in \mathbb{R}_{p,o}^{p\times p}(s)$. Let $\hat R_R = \hat S_R \hat D_R$, i.e. $\hat S_R = \hat R_R \hat D_R^{-1}$.

Since $\hat S_R \hat D_R = \hat N_R - \hat Q_R \hat D_R$, $\hat R_R$ is a polynomial matrix. Further, $\hat S_R$ being strictly proper ~~and $\hat D_R$ column reduced,~~ the column degrees of $\hat R_R$ must be strictly less than those of $\hat D_R$. A similar proof establishes the fact for left fractions. □

6.3.2.2 Interactor Matrix and Hermite Form

Interactor Matrix

In Chapter 3, we observed that SISO model reference adaptive control requires the knowledge of the relative degree of the transfer function $\hat P(s)$. The extension of the concept of relative degree to transfer function matrices is not trivial and must take into account the high-frequency interactions between different inputs and outputs. The concept of an *interactor matrix* follows in a natural way, by taking the following approach. Note that the knowledge of the relative degree of a scalar transfer function $\hat P(s) \in \mathbb{R}_{p,o}(s)$ is equivalent to the knowledge of a monic polynomial $\hat\xi(s)$ such that

$$\lim_{s\to\infty} \hat\xi(s)\hat P(s) = k_p \neq 0$$

Then, the relative degree of $\hat P(s)$ is equal to the degree of $\hat\xi(s)$. The

scalar k_p was earlier called the high-frequency gain of the plant $\hat P$.

The high-frequency behavior of MIMO systems is similarly determined by a polynomial matrix such that

$$\lim_{s\to\infty} \hat\xi(s)\hat P(s) = K_p \quad \text{nonsingular}$$

We must assume here that $\hat P(s)$ is itself nonsingular. The matrix $\hat\xi(s)$ is not unique, unless its structure is somewhat restricted. It is shown in Wolovich & Falb [1976] that there exists a unique matrix $\hat\xi(s)$ satisfying the following conditions.

Definition Interactor Matrix

The *interactor matrix* of a nonsingular plant $\hat P(s) \in \mathbb{R}_p^{p\times p}(s)$ is the unique matrix $\hat\xi \in \mathbb{R}^{p\times p}[s]$ such that

$$\lim_{s\to\infty} \hat\xi(s)\hat P(s) = K_p \quad \text{nonsingular}$$

$$\hat\xi(s) = \hat\Sigma(s)\hat\Delta(s)$$

where

- $\hat\Delta(s) = \text{diag}(s^{r_i})$
- $$\hat\Sigma(s) = \begin{bmatrix} 1 & 0 & 0 & \cdot & 0 \\ \hat\sigma_{21}(s) & 1 & 0 & \cdot & 0 \\ \hat\sigma_{31}(s) & \hat\sigma_{32}(s) & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \hat\sigma_{p1}(s) & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$
- any polynomial $\hat\sigma_{ij}(s)$ is divisible by s (or is zero)

The matrix K_p is called the *high frequency gain* of the plant $\hat P(s)$. The integers r_i extend the notion of the relative degree r of an SISO transfer function. The matrix $\hat\Sigma(s)$, which becomes 1 in the SISO case, describes the high-frequency interconnections between different inputs and outputs.

Hermite Normal Form

Another approach, leading to an equivalent definition, is found in Morse [1976]. For this, one notes that proper rational functions of s form a *ring*. A division algorithm may also be defined, where the *gauge* of an element is its relative degree. Therefore, $\hat P(s)$ is a matrix whose elements belong to a principal ideal domain, and may be factored as

$$\hat{P}(s) = \hat{H}(s) \cdot \hat{U}(s) \qquad \hat{H}, \hat{U} \in \mathbb{R}_p^{p \times p}(s)$$

where $\hat{H}$ is the *Hermite column form* of $\hat{P}$. The Hermite column form of $\hat{H}$ is a lower triangular matrix, such that the elements below the diagonal are either zero, or have relative degree strictly less than the diagonal element on the same row. The matrix $\hat{U}$ is unimodular in $\mathbb{R}_p^{p \times p}(s)$, that is, its inverse is a proper rational matrix. The unimodularity of $\hat{U}$ is equivalent to

$$\lim_{s \to \infty} \hat{U}(s) = K_u \quad \text{nonsingular}$$

Morse [1979] showed that, with some slight modifications, one could *uniquely* define the *Hermite normal form* as follows

Definition Hermite Normal Form

The *Hermite normal form* of a nonsingular plant $\hat{P}(s) \in \mathbb{R}_p^{p \times p}(s)$ is the unique matrix $\hat{H} \in \mathbb{R}_p^{p \times p}(s)$ such that

$$\hat{P}(s) = \hat{H}(s) \cdot \hat{U}(s) \qquad \hat{U} \text{ unimodular in } \mathbb{R}_p^{p \times p}(s)$$

$$\hat{H} = \begin{bmatrix} \frac{1}{(s+a)^{r_1}} & 0 & \cdot & \cdot \\ \frac{\hat{h}_{21}(s)}{(s+a)^{r_2 - 1}} & \frac{1}{(s+a)^{r_2}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{(s+a)^{r_p}} \end{bmatrix}$$

where $\partial \hat{h}_{ij}(s) \le r_i - 1$ and a is arbitrary, but fixed *a priori*.

As shown in the following proposition, the interactor matrix and the Hermite normal form are completely equivalent.

Proposition 6.3.2 Interactor Matrix and Hermite Normal Form Equivalence

Let $\hat{\xi}(s)$ be the interactor matrix of $\hat{P}(s) \in \mathbb{R}_p^{p \times p}(s)$. Let $\hat{H}(s)$ be the Hermite normal form of $\hat{P}(s)$ for $a = 0$.

Then $\hat{\xi}(s) = \hat{H}^{-1}(s)$

Proof of Proposition 6.3.2

We let $\hat{\xi} = \hat{H}^{-1}$ and show that it satisfies the conditions in the definition of the interactor matrix. First, note that $\hat{\xi}\hat{P} = \hat{H}^{-1}\hat{P} = \hat{U}$. Since $\hat{U}$ is unimodular, $\lim_{s \to \infty} \hat{\xi}\hat{P} = K_u$ nonsingular. Next, decompose

$$\hat{H} = \begin{bmatrix} s^{-r_1} & 0 & \cdot & 0 \\ 0 & s^{-r_2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & s^{-r_p} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdot & 0 \\ s\hat{h}_{21}(s) & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ s\hat{h}_{p1}(s) & \cdot & \cdot & \cdot \end{bmatrix}$$

$$= \hat{\Delta}^{-1}\hat{\Sigma}^{-1} = (\hat{\Sigma}\hat{\Delta})^{-1}$$

Clearly, $\hat{\Delta}$ so defined satisfies the required properties, and $\hat{\Sigma}$ is a lower triangular matrix with 1's on the diagonal. The off-diagonal terms satisfy

$$\hat{\sigma}_{21}(s) = -s\hat{h}_{21}(s)$$
$$\hat{\sigma}_{31}(s) = s(\hat{h}_{31}(s) - \hat{h}_{32}(s)\hat{h}_{31}(s))$$
$$\hat{\sigma}_{32}(s) = -s\hat{h}_{32}(s)$$
$$\cdots$$

so that $\hat{\Sigma}(s)$ also satisfies the required conditions. □

Hermite Form and Model Reference Control

The significance of the Hermite normal form in model reference adaptive control may be understood from the following discussion. An arbitrary linear time-invariant (LTI) controller may be represented as

$$u = \hat{C}_{FF}(r) + \hat{C}_{FB}(y_p)$$

where $\hat{C}_{FF}$ is a *feedforward* controller and $\hat{C}_{FB}$ is a *feedback* controller, so that the closed-loop transfer function is given by

$$y_p = (I - \hat{P}\hat{C}_{FB})^{-1}\hat{P}\hat{C}_{FF}(u)$$
$$= \hat{P}(I - \hat{C}_{FB}\hat{P})^{-1}\hat{C}_{FF}(u)$$

$$= \hat{H}\hat{U}(I - \hat{C}_{FB}\hat{P})^{-1}\hat{C}_{FF}(u)$$

The transfer function is equal to the reference model transfer function $\hat{M}$ if

$$\hat{M} = \hat{H}\hat{U}(I - \hat{C}_{FB}\hat{P})^{-1}\hat{C}_{FF}$$

Assume now that the plant is *strictly proper*, and restrict the controller to be proper. Then, the transfer function $\hat{U}(I - \hat{C}_{FB}\hat{P})^{-1}\hat{C}_{FF}$ is proper. In other words, the reference model must be the product of the plant's Hermite form times an (arbitrary) proper transfer function. For SISO systems, this is equivalent to saying that a proper compensator cannot reduce the relative degree of a strictly proper plant.

6.3.3 Model Reference Adaptive Control—Controller Structure

With the foregoing preliminaries, the assumptions required for multi-input multi-output (MIMO) model reference adaptive control will look fairly similar to the assumptions in the SISO case.

Assumptions

(A1) Plant Assumptions
The plant is a strictly proper MIMO LTI system, described by a square, nonsingular, transfer function matrix

$$\hat{P} = \hat{H}\hat{U} \in \mathbb{R}_{p,o}^{p \times p}(s)$$

where $\hat{H}$ is a stable Hermite normal form of $\hat{P}$, obtained by setting $a > 0$. $\hat{H}$ is assumed known. The plant is minimum phase, and the observability index ν is known (an upper bound on the order of the system is, therefore, νp).

(A2) Reference Model Assumptions
The reference model is described by

$$\hat{M} = \hat{H}\hat{M}_0 \in \mathbb{R}_{p,o}^{p \times p}(s)$$

where $\hat{M}_0$ is a proper, stable transfer function matrix and $\hat{H}$ is the Hermite normal form of the plant.

(A3) Reference Input Assumptions
The reference input $r(\cdot) \in \mathbb{R}^m$ is piecewise continuous and bounded on $\mathbb{R}_+$.

Controller Structure

First, note that all the dynamics of $\hat{M}_0$ may be realized by prefiltering the reference input r. Therefore, we define

$$\bar{r} = \hat{M}_0(r)$$

so that

$$y_m = \hat{H}(\bar{r}) \tag{6.3.1}$$

In this manner, the model reference adaptive control problem is replaced by a problem where the reference model $\hat{M}$ is equal to the Hermite normal form of the plant.

The controller structure used for multivariable adaptive control is similar to the SISO structure, provided that adequate transpositions are made. Let $\hat{\lambda}(s)$ be an arbitrary, monic, Hurwitz polynomial of degree $\nu - 1$ (where ν is the observability index of $\hat{P}$). Define $\hat{\Lambda}(s) \in \mathbb{R}^{p \times p}(s)$ such that

$$\hat{\Lambda}(s) = \text{diag}\,[\hat{\lambda}(s)] \tag{6.3.2}$$

The controller is represented in Figure 6.6.

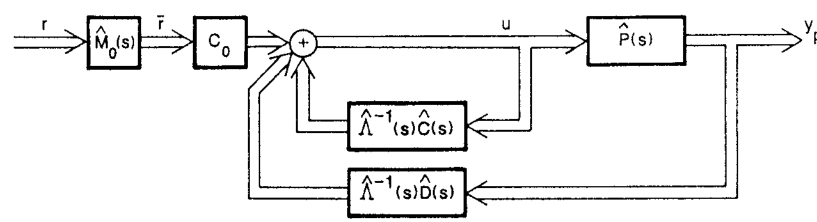

Figure 6.6 MIMO Controller Structure

It is defined by

$$\bar{r} = \hat{M}_0(r)$$

$$u = C_0\bar{r} + \hat{\Lambda}^{-1}(s)\hat{C}(s)(u) + \hat{\Lambda}^{-1}(s)\hat{D}(s)(y_p) \tag{6.3.3}$$

where $C_0 \in \mathbb{R}^{p \times p}$, $\hat{C}(s), \hat{D}(s) \in \mathbb{R}^{p \times p}(s)$. By the foregoing choice of $\hat{\Lambda}$, $\hat{\Lambda}^{-1}\hat{C} = \hat{C}\hat{\Lambda}^{-1}$, and $\hat{\Lambda}^{-1}\hat{D} = \hat{D}\hat{\Lambda}^{-1}$. Further, let $\partial\hat{C} \leq \nu - 2$, $\partial\hat{D} \leq \nu - 1$ (where $\partial\hat{C}$ denotes the maximum degree of all elements of $\hat{C}$).

Now, consider a right coprime fraction $(\hat{N}_R, \hat{D}_R)$ of $\hat{P}$. Therefore

$$\hat{D}_R(x) = u \qquad y_p = \hat{N}_R(x) \tag{6.3.4}$$

where x is a pseudo-state of $\hat{P}$. Combining with the expression of the controller

$$\hat{D}_R(x) = C_0\bar{r} + \hat{\Lambda}^{-1}\hat{C}(u) + \hat{\Lambda}^{-1}\hat{D}(y_p) \tag{6.3.5}$$

so that

$$\hat{\Lambda}\hat{D}_R x = \hat{\Lambda}C_0\bar{r} + \hat{C}\hat{D}_R x + \hat{D}\hat{N}_R x \tag{6.3.6}$$

$$[(\hat{\Lambda} - \hat{C})\hat{D}_R - \hat{D}\hat{N}_R]x = \hat{\Lambda}C_0\bar{r} \tag{6.3.7}$$

Therefore, the output y_p is given by

$$y_p = \hat{N}_R[(\hat{\Lambda} - \hat{C})\hat{D}_R - \hat{D}\hat{N}_R]^{-1}\hat{\Lambda}C_0(\bar{r}) \tag{6.3.8}$$

As in the SISO case, this leads us to a proposition guaranteeing that there exists a nominal controller of *prescribed degree* such that the closed-loop transfer function matches the reference model ($\hat{H}$) transfer function.

Proposition 6.3.3 MIMO Matching Equality

There exist C_0^*, $\hat{C}^*$, $\hat{D}^*$ such that the transfer function from $\bar{r} \to y_p$ is $\hat{H}$.

Proof of Proposition 6.3.3

The transfer function from $\bar{r}$ to y_p is $\hat{H}$ if and only if the following equality is satisfied

$$\hat{N}_R[(\hat{\Lambda} - \hat{C}^*)\hat{D}_R - \hat{D}^*\hat{N}_R]^{-1}\hat{\Lambda}C_0^* = \hat{H} \tag{6.3.9}$$

Since $\hat{H}^{-1}$ is a polynomial matrix, the foregoing equality may be transformed into a polynomial matrix equality, reminiscent of the *matching equality* of Chapter 3

$$(\hat{\Lambda} - \hat{C}^*)\hat{D}_R - \hat{D}^*\hat{N}_R = \hat{\Lambda}C_0^*\hat{H}^{-1}\hat{N}_R \tag{6.3.10}$$

First, we determine C_0^*. Multiply both sides by $\hat{D}_R^{-1}$ on the right and $\hat{\Lambda}^{-1}$ on the left. Then

$$(I - \hat{\Lambda}^{-1}\hat{C}^*) - \hat{\Lambda}^{-1}\hat{D}^*\hat{N}_R\hat{D}_R^{-1} = C_0^*\hat{H}^{-1}\hat{N}_R\hat{D}_R^{-1} \tag{6.3.11}$$

Taking the limit as $s \to \infty$

$$I = C_0^* K_p \to C_0^* = K_p^{-1} \tag{6.3.12}$$

The polynomial matrices $\hat{C}^*$, $\hat{D}^*$ are obtained almost as in the SISO case. Let $(\hat{D}_L, \hat{N}_L)$ be a left coprime fraction of $\hat{P}$, with $\hat{D}_L$ ~~row reduced~~. Divide $\hat{\Lambda}K_p^{-1}\hat{H}^{-1}$ *on the right* by $\hat{D}_L$, so that

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$$\hat{\Lambda}\hat{K}_p^{-1}\hat{H}^{-1} = \hat{Q}_L\hat{D}_L + \hat{R}_L \tag{6.3.13}$$

where $\partial_{ci}(\hat{R}_L) < \partial_{ci}(\hat{D}_L) \le \nu$. Then, let

$$\hat{D}^* = -\hat{R}_L = \hat{Q}_L\hat{D}_L - \hat{\Lambda}K_P^{-1}\hat{H}^{-1}$$
$$\hat{C}^* = \hat{\Lambda} - \hat{Q}_L\hat{N}_L \tag{6.3.14}$$

Since $\hat{D}_L\hat{N}_R = \hat{N}_L\hat{D}_R$, it is easy to show that the given C_0^*, $\hat{C}^*$, $\hat{D}^*$ solve the matching equality. Since $\partial_{ci}(\hat{D}_L) \le \nu$, $\partial\hat{D}^* = \partial(\hat{R}_L) \le \nu - 1$. On the other hand

$$\lim_{s\to\infty}(I - \hat{\Lambda}^{-1}\hat{C}^*) = \lim_{s\to\infty}(\hat{\Lambda}^{-1}\hat{D}^*\hat{P} + K_p^{-1}\hat{H}^{-1}\hat{P}) = I \tag{6.3.15}$$

so that $\hat{\Lambda}^{-1}\hat{C}^*$ is strictly proper and $\partial\hat{C}^* \le \nu - 2$. □

State-Space Representation

A state-space representation is obtained by defining the matrices $C_1, \ldots, C_{\nu-1}, D_0, D_1, \ldots, D_{\nu-1} \in \mathbb{R}^{p\times p}$ such that

$$\hat{C}(s)\hat{\Lambda}^{-1}(s) := C_1\frac{1}{\hat{\lambda}} + C_2\frac{s}{\hat{\lambda}} + \cdots + C_{\nu-1}\frac{s^{\nu-2}}{\hat{\lambda}}$$
$$\hat{D}(s)\hat{\Lambda}^{-1}(s) := D_0 + D_1\frac{1}{\hat{\lambda}} + D_2\frac{s}{\hat{\lambda}} + \cdots + D_{\nu-1}\frac{s^{\nu-2}}{\hat{\lambda}} \tag{6.3.16}$$

Consequently, the vectors $w_i^{(1)}$ and $w_i^{(2)}$ are defined by

$$w_i^{(1)} := \frac{s^{i-1}}{\hat{\lambda}}(u) \qquad w_i^{(2)} := \frac{s^{i-1}}{\hat{\lambda}}(y_p) \quad i = 1\ldots\nu-1 \tag{6.3.17}$$

The regressor vector w is defined as

$$w^T := (\bar{r}^T, w_1^{(1)^T}, \ldots, w_{\nu-1}^{(1)^T}, y_p^T, w_1^{(2)^T}, \ldots, w_{\nu-1}^{(2)^T}) \in \mathbb{R}^{p\times 2\nu p} \tag{6.3.18}$$

and the matrix of controller parameters is

$$\Theta^T := (C_0, C_1, \ldots, C_{\nu-1}, D_0, D_1, \ldots, D_{\nu-1}) \in \mathbb{R}^{p\times 2\nu p} \tag{6.3.19}$$

so that the control input is given by

$$u = \Theta^T w \in \mathbb{R}^p \tag{6.3.20}$$

The controller structure is represented in Figure 6.7. By letting the controller parameter Θ, i.e., $C_0, \ldots, C_{\nu-1}, D_0, \ldots, D_{\nu-1}$ vary with time, the scheme will be made adaptive. We define the parameter error

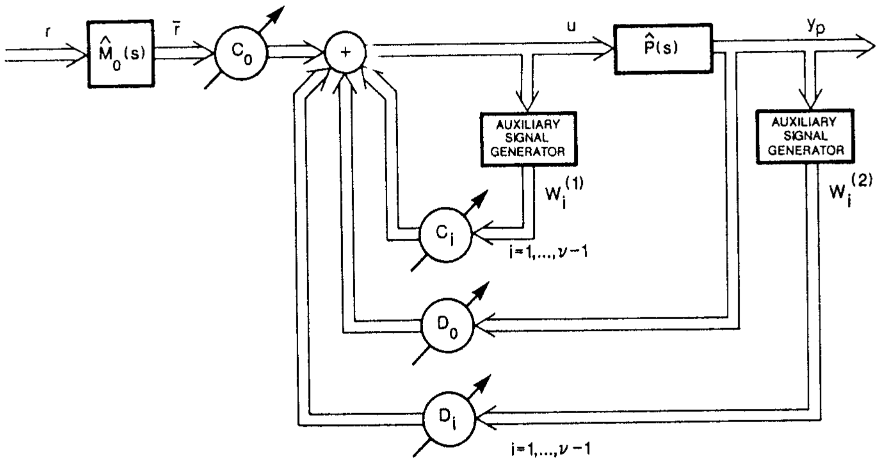

Figure 6.7 MIMO Controller Structure—Adaptive Form

$$\Phi(t) \;:=\; \Theta(t) - \Theta^* \;\in\; \mathbb{R}^{p \times 2\nu p} \tag{6.3.21}$$

and

$$\begin{aligned} \bar\Theta^T &:= (C_1, \ldots, C_{\nu-1}, D_0, D_1, \ldots, D_{\nu-1}) \\ \bar w^T &:= (w_1^{(1)^T}, \ldots, w_{\nu-1}^{(1)^T}, y_p^T, w_1^{(2)^T}, \ldots, w_{\nu-1}^{(2)^T}) \\ \bar\Phi &:= \bar\Theta - \bar\Theta^* \end{aligned} \tag{6.3.22}$$

6.3.4 Model Reference Adaptive Control—Input Error Scheme

For simplicity in the following derivations, we make the following assumption

(A4) High Frequency Gain Assumption
The high-frequency gain matrix K_p is known.

Consequently, we let

$$\begin{aligned} C_0 &= C_0^* = K_p^{-1} \\ u &= C_0^* \bar r + \bar\Theta^T \bar w \end{aligned} \tag{6.3.23}$$

and we look for an update law for $\bar\Theta(t)$.

Consider the matching equality

$$(\hat\Lambda - \hat C^*)\hat D_R - \hat D^* \hat N_R \;=\; \hat\Lambda C_0^* \hat H^{-1} \hat N_R \tag{6.3.24}$$

and multiply both sides by $\hat\Lambda^{-1}$ on the left and $\hat D_R^{-1}$ on the right so that

$$I - (\hat\Lambda^{-1}\hat C^*) - (\hat\Lambda^{-1}\hat D^*)\hat P \;=\; C_0^* \hat H^{-1} \hat P \tag{6.3.25}$$

Now, define

$$\hat L(s) \;:=\; \text{diag}\;[\hat l(s)] \;\in\; \mathbb{R}^{p \times p}(s) \tag{6.3.26}$$

where $\hat l(s)$ is a monic, Hurwitz polynomial such that $\partial \hat l = \partial(\hat H^{-1})$ (the maximum degree of all elements of $\hat H^{-1}$). Since $\hat\Lambda$ and $\hat L$ are given by (6.3.2), (6.3.26), they commute with any matrix. Multiplying both sides of (6.3.25) by $\hat L^{-1}$ and applying both transfer function matrices to u leads to

$$\begin{aligned} \hat L^{-1}(u) &= C_0^*(\hat H \hat L)^{-1}(y_p) + \hat L^{-1}(\hat C^* \hat\Lambda^{-1}(u) + \hat D^* \hat\Lambda^{-1}(y_p)) \\ &= C_0^*(\hat H \hat L)^{-1}(y_p) + \hat L^{-1}(\bar\Theta^{*^T} \bar w) \\ &= C_0^*(\hat H \hat L)^{-1}(y_p) + \bar\Theta^{*^T} \hat L^{-1}(\bar w) \end{aligned} \tag{6.3.27}$$

As in the SISO case, we define

$$\begin{aligned} v^T &:= ((\hat H \hat L)^{-1} y_p^T, \hat L^{-1}(\bar w^T)) \\ \bar v^T &:= \hat L^{-1}(\bar w^T) \\ e_2 &:= \Theta^T v - \hat L^{-1}(u) \end{aligned} \tag{6.3.28}$$

so that, using (6.3.27)

$$\begin{aligned} e_2 &= \Phi^T v \\ &= \bar\Phi^T \bar v \end{aligned} \tag{6.3.29}$$

where the last equality follows because we assumed that K_p is known, that is, $C_0 = C_0^*$. The error equation is a linear equation, but a multivariable one, with $e_2 \in \mathbb{R}^p$, $\bar\Phi^T \in \mathbb{R}^{p \times (2\nu - 1)p}$. However, standard update laws are easily extended to the multivariable case, with similar properties. For example, the normalized gradient algorithm for (6.3.29) becomes

$$\dot{\bar\Theta} \;=\; -g \frac{\bar v e_2^T}{1 + \gamma \bar v^T \bar v} \qquad g, \gamma > 0 \tag{6.3.30}$$

This equation may be obtained by considering each component of e_2, forming p scalar error equations, and collecting the standard SISO update laws. Similarly, a least-squares algorithm may also be defined.

The scheme may also be extended to the unknown high-frequency gain case. However, some procedure must be devised to prevent C_0 from becoming singular (cf. $c_0 \neq 0$ in SISO).

6.3.5 Alternate Schemes

The multivariable input error scheme presented here is equivalent to the scheme presented by Elliott & Wolovich [1982]. In discrete time, a similar scheme is obtained by Goodwin & Long [1980]. An output error version of the model reference adaptive control scheme is found in Singh & Narendra [1984].

It is fairly straight forward to derive an indirect scheme, based on the solution of the matching equality given in the proof of proposition 6.3.3. An interesting contribution of the proposition is to show that the solution of the matching equality requires only one polynomial matrix division. This is to be contrasted with the situation for pole placement, where the general Diophantine equation needs to be solved. In fact, the matrix polynomial division itself is simpler than it looks at first, and may be calculated without matrix polynomial inversion (see Wolovich [1984]).

A possible advantage of the indirect approach is that the interactor matrix may be estimated on-line (Elliott & Wolovich [1984]). Indeed, the requirement of the knowledge of the Hermite form may be too much to ask for, unless $\hat{H}$ is diagonal (cf. Singh & Narendra [1982], [1984]). When $\hat{H}$ is not diagonal, the off-diagonal elements depend on the unknown plant parameters.

The model reference adaptive control objective with $\hat{M} = \hat{H}$ is somewhat restrictive: Indeed, all desired dynamics are generated by *prefiltering* the input signal. In continuous time, all zeros are cancelled by poles, and the remaining poles are defined by $\hat{H}$. In discrete time, the remaining poles are all at the origin (d-step ahead control). This is not very desirable for implementation, as discussed in Chapter 3. In the SISO case, a more adequate scheme was presented such that the internal dynamics of the closed-loop system are actually those of the reference model.

Stability proofs for MIMO adaptive control follow similar paths as for SISO (Goodwin & Long [1980], Singh & Narendra [1982]). Convergence properties have not been established. Indeed, the *uniqueness* of the controller parameter is not guaranteed by proposition 6.3.3, as it was in the SISO case. More research will be needed for the dynamics of

MIMO adaptive control systems to be well-understood.

6.4 CONCLUSIONS

In this chapter, we first discussed how prior information may be used in the context of identification. Then, we presented flexible indirect adaptive control schemes, and proved their global stability under a richness condition on the exogeneous reference input. Applications to pole placement and the factorization approach were discussed. We then turned to the extension of model reference adaptive control schemes to MIMO systems. After some preliminaries, we established a parameterization of the adaptive controller, following lines parallel to the SISO case. An input error scheme was finally presented. More research will be needed to better understand the dynamics of MIMO adaptive control schemes. The combination of modern control theories based on factorization approaches and of MIMO recursive identification algorithms in a flexible indirect adaptive controller scheme is a promising area for further developments and applications.