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Subanalytic Geometry

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Abstract. Lou van den Dries has suggested that the *o*-minimal structure of the classes of semialgebraic or subanalytic sets makes precise Grothendieck's idea of a "tame topology" based on stratification. These notes present another viewpoint (with intriguing possible relationships): we describe a range of classes of spaces between semialgebraic and subanalytic, that do not necessarily fit into the *o*-minimal framework, but that are "tame" from algebraic or analytic perspectives.

1. Introduction

Semialgebraic and subanalytic sets capture ideas in several areas: In model theory, they express properties of quantifier elimination. In geometry and analysis, they provide a language for questions about the local behaviour of algebraic and analytic mappings. Lou van den Dries has suggested that the *o*-minimal structure of the classes of semialgebraic or subanalytic sets makes precise Grothendieck's vision of a "tame topology". (In his provocative *Esquisse d'un programme*, Grothendieck [1984] proposes an axiomatic development of a topology based on ideas of stratification in order to study, for example, singularities that arise in compactifications of moduli spaces.) These notes present another point of view (with intriguing possible relationships): we will describe a range of geometric classes of spaces between semialgebraic and subanalytic, that do not necessarily fit into the *o*-minimal framework, but that are "tame" from algebraic or analytic perspectives. The questions we discuss are in directions pioneered by Whitney, Thom, Łojasiewicz, Gabrielov and Hironaka. (We will

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not try to give a general survey of recent results in the area of semialgebraic and subanalytic sets.)

Semialgebraic and semianalytic sets. A *semialgebraic subset of* $\mathbb{R}^n$ is a subset of the form

$$X = \bigcup_{i=1}^{p} \bigcap_{j=1}^{q} X_{ij}, \tag{1.1}$$

where each X_{ij} is of the form $\{f_{ij}(x) = 0\}$ or $\{f_{ij}(x) > 0\}$, with $f_{ij}(x) = f_{ij}(x_1, \ldots, x_n)$ a polynomial. For example, Figure 1 shows an *algebraic* subset X of $\mathbb{R}^3$ defined by the equation $z^2 - xy^2 = 0$ ("Whitney's umbrella"). Figure 1 illustrates a stratification of X. A *stratification* means a finite (or locally finite) partition into connected smooth manifolds (*strata*) within the class (here, semialgebraic), such that the frontier of each stratum is a union of strata of lower dimension.

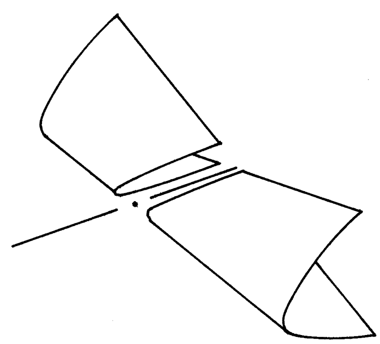

Figure 1. Stratification of Whitney's umbrella $z^2 - xy^2 = 0$.

According to the *Tarski–Seidenberg theorem*, the image of a semialgebraic subset X of $\mathbb{R}^{m+n}$ by a projection $\mathbb{R}^{m+n} \to \mathbb{R}^n$ is semialgebraic. (For the basic properties of semialgebraic or subanalytic sets, see [Bierstone and Milman 1988], for example.) The Tarski–Seidenberg theorem is an assertion about *elimination of quantifiers*; it says that any formula obtained using a finite number of "and", "or", negations, existential and universal quantifiers, from formulas of the form $f(x) = 0$ or $f(x) > 0$, where $f(x) = f(x_1, \ldots, x_n)$ is a polynomial, describes the same set as a formula without quantifiers.

A *semianalytic* subset of $\mathbb{R}^n$ is a subset that is defined *locally* (i.e., in some neighbourhood of any point of $\mathbb{R}^n$) by an expression of the form (1.1), but where the functions f_{ij} are real-analytic. A projection of even a compact semianalytic set need not be semianalytic. (See Example 2.3 below.)

Subanalytic sets. A subset X of $\mathbb{R}^n$ is *subanalytic* if, locally, X is a projection of a relatively compact semianalytic set. The class of semianalytic sets was enlarged to include projections in this way by Łojasiewicz [1964], although the term “subanalytic” is due to Hironaka [1973]. Gabrielov’s *theorem of the complement* asserts that the complement of any subanalytic set is subanalytic [Gabrielov 1968]. This is also an assertion about “quantifier simplification”: The complement of a subanalytic set is defined locally by a formula involving real-analytic equations and inequalities, with existential and universal quantifiers; Gabrielov’s theorem says that the complement can be defined locally by an existential formula.

The uniformization theorem. A closed subanalytic subset X of $\mathbb{R}^n$ is locally a projection of a compact semianalytic set. In fact, closed subanalytic sets are precisely the images of proper real-analytic mappings (from manifolds), according to the following *uniformization theorem*:

THEOREM 1.2. *Let X be a closed subanalytic subset of $\mathbb{R}^n$. Then X is the image of a proper real-analytic mapping φ: $M \to \mathbb{R}^n$, where M is a real-analytic manifold of the same dimension as X.*

The uniformization theorem is a consequence of resolution of singularities [Hironaka 1964; 1974; Bierstone and Milman 1997]; but see [Bierstone and Milman 1988, Sections 4 and 5] for a short elementary proof. From the point of view of the uniformization theorem, subanalytic sets can be viewed as real analogues of complex-analytic sets, or analytic analogues of semialgebraic sets. (See the table *Images of proper mappings* on the next page.) These classes share many properties of a “tame topology”. For example, any set in the class (locally) has finitely many connected components, each in the class; the components, boundary and interior of a set in the class are in the class; a set in the class can be stratified or even triangulated by subsets in the class.

But there are crucial distinctions between the behaviour of semialgebraic and general subanalytic sets. An important example that we will not deal with explicitly is their behaviour at infinity: A semialgebraic subset of $\mathbb{R}^n$ remains semialgebraic at infinity (i.e., when $\mathbb{R}^n$ is compactified to real projective space $\mathbb{P}^n(\mathbb{R})$). This is false for subanalytic sets, in general. An understanding of the behaviour at infinity of certain important classes of subanalytic sets (as in [Wilkie 1996]) represents the most striking success of the model-theoretic point of view in subanalytic geometry.

Grothendieck, in *Esquisse d’un programme*, suggests that “tame” should reflect not only conditions on strata, but also the way that the strata fit together. (For example, semialgebraic and subanalytic sets admit stratifications that are *Lipschitz locally trivial* [Mostowski 1985; Parusiński 1994]). The way that strata are attached to each other is closely related to the way that the local behaviour of X varies along a given stratum S, or as we approach $\bar{S} \setminus S$. Grothendieck envisaged a hierarchy of tame geometric categories from semialgebraic to subanalytic.

Section 2 below includes a sequence of examples illustrating differences in the local behaviour of semialgebraic and general subanalytic sets. The results described in the following sections are directed towards understanding these phenomena; Theorems 3.1 and 4.4 characterize certain subclasses of subanalytic sets that are tame from algebraic or or analytic perspectives, although they do not necessarily fit into an *o*-minimal framework. The uniformization theorem provides the point of view toward subanalytic geometry that is taken here: On the one hand, subanalytic sets provide a natural language for questions about the local behaviour of analytic mappings, and, on the other, local invariants of analytic mappings can be used to characterize a hierarchy of "tame" classes of sets (*Nash-subanalytic*, *semicoherent*, ...) intermediate between semialgebraic and subanalytic.

The phenomena studied in these notes concern not peculiarities of the reals, but rather the local behaviour of analytic mappings whether real or complex. Although it is true, for example, that the image of an arbitrary complex-analytic mapping is a closed analytic set (by the theorem of Remmert [1957]), a complex-analytic set X can be realized, more precisely, as the image of a proper complex-analytic mapping φ that is *relatively algebraic* over any sufficiently small open subset V of the target; this means there is a closed embedding $\iota : \varphi^{-1}(V) \to V \times \mathbb{P}^k(\mathbb{C})$ commuting with the projections to V, whose image is defined by homogeneous polynomial equations (in terms of the homogeneous coordinates of $\mathbb{P}^k(\mathbb{C})$) with coefficients analytic functions on V (by resolution of singularities [Hironaka 1964; 1974; Bierstone and Milman 1997]). The image of a proper real-analytic mapping satisfying the analogous condition is semianalytic, by Łojasiewicz's generalization of the Tarski–Seidenberg theorem [Łojasiewicz 1964; Bierstone and Milman 1988, Theorem 2.2]. Subanalytic sets, on the other hand, provide a natural setting for questions about the local behaviour of analytic mappings in general.

Images of proper mappings.

	Algebraic	Relatively algebraic	Analytic
$\mathbb{C}$	closed algebraic sets	closed analytic sets $\longrightarrow$	
$\mathbb{R}$	closed semialgebraic sets	closed semianalytic sets	closed subanalytic sets

(In the real case, "proper" imposes no restriction on local behaviour.)

Uniformization and rectilinearization. The uniformization theorem above is closely related to the following *rectilinearization theorem* for subanalytic functions. Let N denote a real-analytic manifold; e.g., $\mathbb{R}^n$. (A function $f\colon X \to \mathbb{R}$, where $X \subset N$, is called *subanalytic* if the graph of f is subanalytic as a subset of $N \times \mathbb{R}$.)

THEOREM 1.3 [Bierstone and Milman 1988, § 5]. *Let* $f\colon N \to \mathbb{R}$ *be a continuous subanalytic function. Then there is a proper analytic surjection* $\varphi\colon M \to N$, *where* $\dim M = \dim N$, *such that* $f \circ \varphi$ *is analytic and locally has only* normal crossings.

The latter condition means that each point of M admits a neighbourhood with a coordinate system $x = (x_1, \ldots, x_n)$ in which $f(\varphi(x)) = x_1^{\alpha_1} \cdots x_n^{\alpha_n} u(x)$ and $u(x)$ does not vanish.

It may be interesting to ask whether the uniformization and rectilinearization properties of semialgebraic or subanalytic sets have reasonable analogues for a given o-minimal structure (or *geometric category* in the sense of [van den Dries and Miller 1996]). This is true, for example, for *restricted subpfaffian sets* (projections of relatively compact semianalytic sets that are defined using Pfaffian functions in the sense of [Khovanskiĭ 1991]). The point is that [Bierstone and Milman 1988, § 4], on "transforming an analytic function to normal crossings by blowings-up", preserves subalgebras of analytic functions that are closed under composition by polynomial mappings, differentiation and division (when the quotient is analytic).

2. Examples

In this section, we describe a range of phenomena that distinguish between the behaviour or algebraic and general analytic mappings. Given a subset X of $\mathbb{R}^n$ (or $\mathbb{C}^n$), let $\mathcal{A}_a(X)$ denote the ideal of germs of analytic functions at a that vanish on X.

Coherence. Every complex-analytic set is *coherent* (according to the theory of Oka and Cartan). This means that, if X is a closed complex-analytic subset of $\mathbb{C}^n$ and $a \in X$, then $\mathcal{A}_a(X)$ generates $\mathcal{A}_b(X)$, for all $b \in X$ in some neighbourhood of a. (See, for example, [Łojasiewicz 1991, § VI.1].)

Real-algebraic sets already need not be coherent [Whitney 1965]; for example, Whitney's umbrella (Figure 1) is not coherent at the origin. Here are two more examples:

EXAMPLE 2.1 [Hironaka 1974]. Let X be the closed algebraic subset of $\mathbb{R}^3$ defined by $z^3 - x^2y^3 = 0$ (Figure 2).

In this example, $\mathcal{A}_0(X) = (z^3 - x^2y^3)$, the ideal of germs of real-analytic functions at 0 generated by $z^3 - x^2y^3$, but, at a nonzero point b of the x-axis,

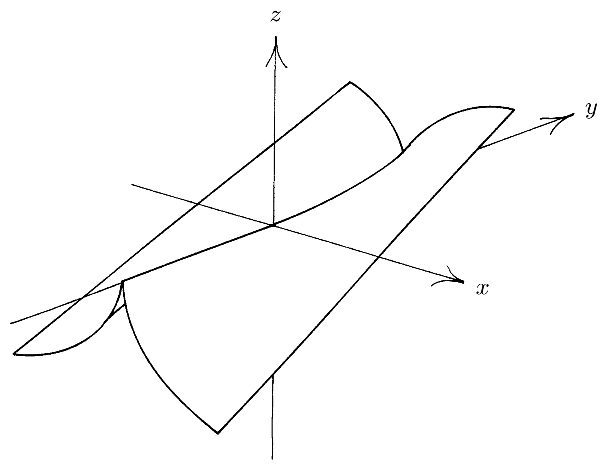

Figure 2. $z^3 - x^2y^3 = 0$.

$z^3 - x^2y^3$ factors non-trivially,

$$z^3 - x^2y^3 = (z - x^{2/3}y)(z^2 + x^{2/3}yz + x^{4/3}y^2),$$

and $\mathcal{A}_b(X) = (z - x^{2/3}y)$. The analytic function $t = z - x^{2/3}y$ defined for $x \neq 0$, is called a *Nash function*: t satisfies a nontrivial polynomial equation $P(x, y, z, t) = 0$. (We can take $P = (t - z)^3 + x^2y^3$.)

Example 2.2 [Bierstone and Milman 1988]. Let X be the real-algebraic subset $z^3 - x^2yz - x^4 = 0$ of $\mathbb{R}^3$ (Figure 3).

The *singularities* of X form a half-line $\{x = z = 0,\ y \geq 0\}$. In particular, $z^3 - x^2yz - x^4$ does not generate $\mathcal{A}_b(X)$, $b \in \{x = z = 0,\ y < 0\}$, so that X is not coherent. In fact, over $\{(x, y) :\ y < 0\}$, $z^3 - x^2yz - x^4 = 0$ can be solved uniquely as $z = g(x, y)$, where g is analytic. This can be seen by transforming the given equation by the quadratic mapping

$$\sigma:\ x = u,\ y = v,\ z = uw.$$

(σ is the *blowing-up* of $\mathbb{R}^3$ with *centre* $\{x = z = 0\}$ (restricted to a local coordinate chart)). Then $\sigma^{-1}(X)$ is given by $u^3(w^3 - vw - u) = 0$; i.e., $\sigma^{-1}(X) = E' \cup X'$, where E' is the coordinate plane $\{u = 0\}$ and X' is the smooth hypersurface $\{u = w^3 - vw\}$, which is transverse to E' when $v < 0$.

Local dimensions of a subanalytic set. At each point of a subanalytic set, we can consider its local topological dimension, as well as the dimensions of its local analytic and formal closures (in a sense we will make precise below). These three local dimensions coincide for analytic or semianalytic sets, but they may all be distinct for subanalytic sets in general [Gabrielov 1971]. Gabrielov's construction (Example 2.5) is based on the following classical example of Osgood (1920's).

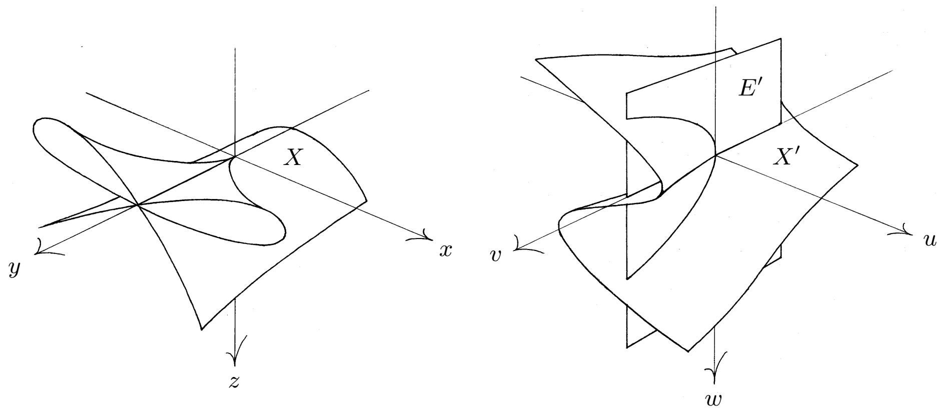

Figure 3. Blowing-up of $X = \{z^3 - x^2yz - x^4 = 0\}$.

Example 2.3. Let φ denote the analytic mapping

$$\varphi(x_1, x_2) = (x_1,\, x_1x_2,\, x_1x_2e^{x_2}).$$

Then there are no (nonzero) formal relations (among the components of φ) at the origin $(x_1, x_2) = (0, 0)$; i.e., if $G(y_1, y_2, y_3)$ is a nonzero formal power series and $G(x_1,\, x_1x_2,\, x_1x_2e^{x_2}) = 0$, then $G = 0$. Indeed, writing $G = \sum_{j=0}^{\infty} G_j$, where $G_j(y_1, y_2, y_3)$ denotes the homogeneous part of G of order k, we have

$$0 = G(x_1,\, x_1x_2,\, x_1x_2e^{x_2}) = \sum_{j=0}^{\infty} x_1^j G_j(1,\, x_2,\, x_2e^{x_2}),$$

so that all $G_j(1,\, x_2,\, x_2e^{x_2}) = 0$, and therefore all polynomials G_j are zero because e^{x_2} is transcendental.

Relations. If $y = \varphi(x)$, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, is a mapping (in some given class), we let φ^* denoted the homomorphism of rings of functions given by composition with φ; i.e., $\varphi^* :\ g(y) \mapsto (g \circ \varphi)(x)$. Then

$$\operatorname{Ker} \varphi^* = \{g(y_1, \dots, y_n) :\ g\big(\varphi_1(x), \dots, \varphi_n(x)\big) = 0\}$$

is, by definition, the ideal of *relations* among the components $\varphi_1, \dots, \varphi_n$ of φ.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let U be an open subset of $\mathbb{K}^m$ and let $a \in U$. We write $\mathbb{K}\{x - a\} = \mathbb{K}\{x_1 - a_1,\, \dots,\, x_m - a_m\}$ or $\mathbb{K}[\![x - a]\!] = \mathbb{K}[\![x_1 - a_1,\, \dots,\, x_m - a_m]\!]$ for the rings of convergent or formal power series (respectively) centred at a. The ring $\mathbb{K}\{x - a\}$ can be identified with the local ring $\mathcal{O}_a$ of germs of analytic functions at a, and $\mathbb{K}[\![x - a]\!]$ with the completion $\widehat{\mathcal{O}}_a$ of $\mathcal{O}_a$. (These notions and remarks make sense, more generally, using local coordinates on an m-dimensional $\mathbb{K}$-analytic manifold U.) We will write $\underline{m}_a$ or $\underline{\hat{m}}_a$ for the maximal ideal of $\mathcal{O}_a$ or $\widehat{\mathcal{O}}_a$ (respectively). Suppose that φ is an analytic mapping $\varphi : U \to \mathbb{K}^n$, and let

$b = \varphi(a)$. Then φ induces ring homomorphisms

$$\varphi_a^* : \ \mathcal{O}_b \to \mathcal{O}_a,$$
$$\hat{\varphi}_a^* : \ \widehat{\mathcal{O}}_b \to \widehat{\mathcal{O}}_a;$$

φ_a^* or $\hat{\varphi}_a^*$ corresponds to composition of convergent or formal power series centred at b (respectively) with the Taylor expansion $\hat{\varphi}_a$ of φ at a. $\operatorname{Ker}\varphi_a^*$ or $\operatorname{Ker}\hat{\varphi}_a^*$ is the ideal of convergent or formal relations (respectively) among the Taylor expansions $\hat{\varphi}_{1,a}, \dots, \hat{\varphi}_{n,a}$ of the components of φ.

If X is an analytic subset of $\mathbb{K}^n$ and $b \in X$, then

$$\dim_b X = \dim \frac{\mathcal{O}_b}{\mathcal{A}_b(X)},$$

where $\dim_b X$ denotes the geometric dimension of X at a, and $\dim \mathcal{O}_b/\mathcal{A}_b(X)$ denotes the *Krull dimension* of the local ring $\mathcal{O}_b/\mathcal{A}_b(X)$; i.e., the length of a longest chain of prime ideals in this ring. (See [Łojasiewicz 1991, § IV.4.3].) Let φ be an analytic mapping and $b = \varphi(a)$, as above. Then there is a smallest (germ of an) analytic set Y_b at b, such that Y_b contains $\varphi(V)$, for a sufficiently small neighbourhood V of a. Clearly,

$$\mathcal{A}_b(Y_b) = \operatorname{Ker}\varphi_a^*.$$

Ranks of Gabrielov. Gabrielov introduced the following three *ranks* associated to an analytic mapping φ at a point a of its source:

$$r_a(\varphi) := \text{generic rank of } \varphi \text{ at } a,$$
$$r_a^{\mathcal{F}}(\varphi) := \dim \frac{\mathbb{K}[\![y - \varphi(a)]\!]}{\operatorname{Ker}\hat{\varphi}_a^*},$$
$$r_a^{\mathcal{A}}(\varphi) := \dim \frac{\mathbb{K}\{y - \varphi(a)\}}{\operatorname{Ker}\varphi_a^*}.$$

(The "generic rank" of φ at a is the largest rank of the tangent mapping of φ in a small neighbourhood of a.) It is not difficult to see that

$$r_a(\varphi) \ \le \ r_a^{\mathcal{F}}(\varphi) \ \le \ r_a^{\mathcal{A}}(\varphi).$$

(We have $r_a^{\mathcal{F}}(\varphi) \le r_a^{\mathcal{A}}(\varphi)$ because $\operatorname{Ker}\varphi_a^* \subset \operatorname{Ker}\hat{\varphi}_a^*$. On the other hand, $r_x(\varphi)$ is constant in a neighbourhood of a, and at a point x where $r_x(\varphi)$ equals the rank of the tangent mapping of φ, all three ranks of Gabrielov coincide (by the implicit function theorem) and $r_x^{\mathcal{F}}(\varphi) \le r_a^{\mathcal{F}}(\varphi)$ (for example, by [Bierstone and Milman 1987a, Prop. 8.3.7]). Therefore, $r_a(\varphi) \le r_a^{\mathcal{F}}(\varphi)$. See also [Milman 1978].)

In the 1960's, Artin and Grothendieck asked: Is $\operatorname{Ker}\hat{\varphi}_a^*$ generated by $\operatorname{Ker}\varphi_a^*$? In other words, is $r_a^{\mathcal{F}}(\varphi) = r_a^{\mathcal{A}}(\varphi)$? Gabrielov [1971] showed that the answer is "no". (See Example 2.5 below.) Of course, if φ is a proper complex-analytic mapping, then all three ranks above coincide, by Remmert's proper mapping theorem [Remmert 1957].

DEFINITION 2.4. We say that φ is *regular at a* if $r_a(\varphi) = r_a^{\mathcal{A}}(\varphi)$. We say that φ is *regular* if it is regular at every point of the source.

Local dimensions. The ranks of Gabrielov have counterparts for a subanalytic set. Let X denote a closed subanalytic subset of $\mathbb{R}^n$. If $b \in X$, then, by definition, $\mathcal{A}_b(X) = \mathcal{A}_b(Y_b)$, where Y_b is the smallest germ of an analytic set at b containing the germ of X. Suppose that $\varphi : M \to \mathbb{R}^n$ is a proper real-analytic mapping from a manifold M, such that $\varphi(M) = X$. Clearly,

$$\mathcal{A}_b(X) = \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \varphi_a^*.$$

This suggests that we define the *formal local ideal* $\mathcal{F}_b(X)$ of X at b as

$$\mathcal{F}_b(X) = \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \hat{\varphi}_a^*.$$

(The preceding intersections are finite: $\operatorname{Ker} \varphi_a^*$ and $\operatorname{Ker} \hat{\varphi}_a^*$ are constant on connected components of the fibre $\varphi^{-1}(b)$ [Bierstone and Milman 1998, Lemma 5.1].) The ideal $\mathcal{F}_b(X)$ does not depend on the mapping φ: There are equivalent ways to define it using X alone [Bierstone and Milman 1998, Lemma 6.1]; for example, $\mathcal{F}_b(X) = \{G \in \widehat{\mathcal{O}}_b : (G \circ \gamma)(t) \equiv 0$ for every real-analytic arc $\gamma(t)$ in X such that $\gamma(0) = b\}$.

We define

$$\begin{aligned} d_b(X) &:= \dim_b X, \\ d_b^{\mathcal{A}}(X) &:= \dim Y_b = \dim \frac{\mathcal{O}_b}{\mathcal{A}_b(X)}, \\ d_b^{\mathcal{F}}(X) &:= \dim \frac{\widehat{\mathcal{O}}_b}{\mathcal{F}_b(X)}. \end{aligned}$$

Then

$$d_b(X) \;\le\; d_b^{\mathcal{F}}(X) \;\le\; d_b^{\mathcal{A}}(X),$$

by the corresponding inequalities among the ranks of Gabrielov. If X is semianalytic, then $\mathcal{F}_b(X)$ is generated by $\mathcal{A}_b(X)$, and all three local dimensions coincide [Łojasiewicz 1964; Bierstone and Milman 1988, Theorem 2.13].

EXAMPLE 2.5 [Gabrielov 1971]. Let $\varphi(x) = \big(\varphi_1(x), \varphi_2(x), \varphi_3(x)\big)$ denote Osgood's mapping (Example 2.3). Then there is a *divergent* power series $G(y) = G(y_1, y_2, y_3)$ such that $\varphi_4(x) := G\big(\varphi(x)\big)$ *converges*: Write

$$\varphi_3(x) = x_1x_2 + x_1x_2^2 + \frac{x_1x_2^3}{2!} + \cdots.$$

We construct a sequence of polynomials $g_k(y)$, for $k = 1, 2, \ldots$, to kill terms of higher and higher order in the expansion of $\varphi_3(x)$:

$$
\begin{aligned}
g_1(y) &:= y_3 - y_2, & g_1 \circ \varphi &= x_1 x_2^2 + \cdots, \\
g_2(y) &:= 2\big(y_1(y_3 - y_2) - y_2^2\big), & g_2 \circ \varphi &= x_1^2 x_2^3 + \cdots, \\
&\ \ \vdots & &\ \ \vdots \\
g_k(y) &:= k\big(y_1 g_{k-1}(y) - y_2^k\big), & g_k \circ \varphi &= x_1^k x_2^{k+1} + \cdots.
\end{aligned}
$$

Note that the maximum absolute value of a coefficient of $g_k(y)$ is $k!$, while the maximum absolute value of a coefficient in the power series expansion of $g_k\big(\varphi(x)\big)$ is 1. It follows that the power series $G(y) := \sum_{k=1}^{\infty} g_k(y)$ diverges, while $\varphi_4(x) := G\big(\varphi(x)\big)$ converges.

Set $\psi(x) := \big(\varphi_1(x), \varphi_2(x), \varphi_3(x), \varphi_4(x)\big)$. It is not difficult to see that

$$
r_0(\psi) = 2, \quad r_0^{\mathcal{F}}(\psi) = 3, \quad r_0^{\mathcal{A}}(\psi) = 4.
$$

The images of proper real-analytic mappings that are regular form an important subclass of closed subanalytic sets, called *Nash-subanalytic*. Regular mappings (and therefore Nash-subanalytic sets) are characterized by a theorem of Gabrielov described in Section 3 below. Although none of Examples 2.1, 2.2, 2.3 or 2.5 above is coherent, it is clear that each satisfies a stratified version of the property of coherence, in some reasonable sense. (For example, the images of Osgood's and Gabrielov's mappings — Examples 2.3 and 2.5 — are coherent outside the origin.) We will describe a larger class of "semicoherent" subanalytic sets that captures such an idea. In Section 4, we will see that semicoherent sets are "tame" from an analytic viewpoint.

Images of proper mappings.

	Algebraic	Relatively algebraic	Regular	
$\mathbb{C}$	closed algebraic sets	closed analytic sets	→	→
$\mathbb{R}$	closed semialgebraic sets	closed semianalytic sets	closed Nash-subanalytic sets	semicoherent sets

Semicoherence. We will say that a subanalytic set X is semicoherent if it has a stratification such that the formal local ideals $\mathcal{F}_b(X)$ of X are generated over each stratum by finitely many subanalytically parametrized formal power series. More precisely:

Definition 2.6. Let $X \supset Z$ denote closed subanalytic subsets of $\mathbb{R}^n$. We say that X is *(formally) semicoherent rel Z (relative to Z)* if X has a (locally finite) subanalytic stratification $X = \bigcup X_i$ such that Z is a union of strata and X satisfies the following *formal semicoherence* property *along* every stratum X_i outside Z: For every point of $\overline{X}_i$, there is a neighbourhood V and there are finitely many formal power series

$$f_{ij}(\,\cdot\,, Y) = \sum_{\alpha\in\mathbb{N}^n} f_{ij,\alpha}(\,\cdot\,)Y_1^{\alpha_1}\cdots Y_n^{\alpha_n}$$

whose coefficients $f_{ij,\alpha}$ are analytic functions on $X_i \cap V$ that are subanalytic (i.e., their graphs are subanalytic as subsets of $V \times \mathbb{R}$), such that, for all $b \in X_i \cap V$, $\mathcal{F}_b(X)$ is generated by the elements $\sum_\alpha f_{ij,\alpha}(b)(y-b)^\alpha \in \mathbb{R}[\![y-b]\!]$. ($(y-b)^\alpha$ means $(y_1-b_1)^{\alpha_1}\cdots(y_n-b_n)^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$.)

Subanalyticity of the coefficients, $f_{ij,\alpha}$ is a natural restriction on their growth at the boundary of a stratum (as expressed by a *Łojasiewicz inequality*; compare [Łojasiewicz 1964; Bierstone and Milman 1988, § 6]). We can formulate an (analytic) semicoherence condition analogous to that above by using the ideals $\mathcal{A}_b(X)$ in place of $\mathcal{F}_b(X)$, but the formal condition seems to be the more useful (as in Theorem 4.4 below). The formal and analytic semicoherence conditions are equivalent if each $\mathcal{F}_b(X)$ is generated by $\mathcal{A}_b(X)$ (as in the case of real- or complex-analytic sets). Semicoherence of semi-algebraic sets was proved by Tougeron and Merrien [Merrien 1980], and of Nash-subanalytic sets by Bierstone and Milman [1987a; 1987b]; in these cases, each $\mathcal{F}_b(X)$ is generated by $\mathcal{A}_b(X)$.

Pawłucki has proved that, if Z denotes the set of *non-Nash points* of X (i.e., the points that do not admit Nash-subanalytic neighbourhoods in X), then Z is subanalytic [1990] and X is semicoherent rel Z [1992]. Pawłucki's theorem implies the analogous statement for the non-semianalytic points of X.

Question. Pawłucki's result suggests the following general question about o-minimal structures on $\mathbb{R}$ (probably easier than Pawłucki's theorem; Nash-subanalytic sets do not correspond to an o-minimal structure): Let $\mathcal{S}_1 \subset \mathcal{S}_2$ be o-minimal structures on $\mathbb{R}$. If X is $\mathcal{S}_2$-definable and $Y \subset X$ denotes the points of X that do not admit $\mathcal{S}_1$-definable neighbourhoods, then is Y $\mathcal{S}_2$-definable? (Piękosz [1998] gives a result in this direction.)

In 1986, Hironaka announced that every subanalytic set X is both $\mathcal{F}$- and $\mathcal{A}$-semicoherent (and, as a consequence, that X admits a subanalytic stratification with the local dimensions $d_b^{\mathcal{F}}(X)$ and $d_b^{\mathcal{A}}(X)$ constant on every stratum) [Hironaka 1986]. But Pawłucki has given a counterexample!

Example 2.7 [Pawłucki 1989]. Let $\{a_n\}$ be any sequence of points in an open interval $I = (-\delta, \delta)$ of $\mathbb{R}$ (where $\delta > 0$). Pawłucki constructs an analytic mapping $\varphi\colon I^3 \to \mathbb{R}^5$ of the form

$$\varphi(u,w,t) = \big(u,\, t,\, tw,\, t\Phi(u,w),\, t\Psi(u,w,t)\big)$$

such that, in $I = I \times \{0\} \times \{0\}$, φ admits no nonzero formal relation (i.e., $\operatorname{Ker}\hat{\varphi}_a^* = 0$) precisely at the points a of $\{a_n\}$, but φ has a nonzero convergent relation (i.e., $\operatorname{Ker}\varphi_a^* \neq 0$) throughout any open interval in $I \setminus \{a_n\}$.

Pawłucki's idea is based on Gabrielov's example 2.5. Take $\delta = \frac{1}{2}$. We can define

$$\Phi(u,w) := \sum_{n=1}^{\infty} \big((u-a_1)\cdots(u-a_n)\big)^{r(n)} w^n,$$

where $\{r(n)\}$ is an increasing sequence of positive integers with $\limsup r(n)/n = \infty$. Write $p_n(u) := \big((u-a_1)\cdots(u-a_n)\big)^{r(n)}$. We define a sequence of rational functions $f_n(u,t,x,y)$ by

$$f_1(u,t,x,y) := \frac{y}{p_1(u)}$$

and, for $n > 1$,

$$\begin{aligned} f_n(u,t,x,y) &:= \frac{p_{n-1}(u)}{p_n(u)} t\big(f_{n-1}(u,t,x,y) - x^{n-1}\big) \\ &= \frac{t^{n-1}y}{p_n(u)} - \frac{1}{p_n(u)} \sum_{k=1}^{n-1} p_k(u) t^{n-k} x^k. \end{aligned}$$

Then each

$$f_n\big(u,\, t,\, tw,\, t\Phi(u,w)\big) = t^n \sum_{k=n}^{\infty} \frac{p_k(u)}{p_n(u)} w^k,$$

and we can set

$$t\Psi(u,w,t) := \sum_{n=1}^{\infty} f_n\big(u,\, t,\, tw,\, t\Phi(u,w)\big).$$

(See [Pawłucki 1989] for details.)

Pawłucki's construction provides examples of a variety of interesting phenomena, depending on the choice of the sequence $\{a_n\}$; for instance:

(1) If $\lim_{n\to\infty} a_n = 0$ but $a_n \neq 0$ for all n, then there is no (nonzero) relation at each a_n, a divergent relation (but no convergent relation) at 0, and a convergent relation at any other point of I. Suppose that $X = \varphi(K)$, where K is a compact subanalytic neighbourhood of 0 in I^3. Clearly, X is neither $\mathcal{F}$- nor $\mathcal{A}$-semicoherent.

(2) If $\{a_n\}$ is dense in I, then there is no convergent relation at any point of I, but there is a formal relation at every point of $I \setminus \{a_n\}$. Therefore, $\mathcal{A}$-semicoherent $\not\Rightarrow$ $\mathcal{F}$-semicoherent.

We believe it is not known whether $\mathcal{F}$-semicoherent $\Rightarrow$ $\mathcal{A}$-semicoherent.

(3) If the accumulation points of $\{a_n\}$ themselves form a convergent sequence $\{c_k\}$, then $X = \varphi(K)$ is not semicoherent precisely at the points $\varphi(c_k)$ and $\varphi(\lim c_k)$ (i.e., these points do not admit semicoherent neighbourhoods in X). In other words, the points at which X is not semicoherent do not necessarily form a subanalytic subset!

The phenomena above show that subanalytic sets in general can be wild indeed. The class of semicoherent sets, on the other hand, can be characterized by several (remarkably equivalent) "tameness" properties (to be described in Section 4). For example, let X be a closed subanalytic subset of $\mathbb{R}^n$, and let $\mathcal{C}^k(X)$ denote the ring of restrictions to X of $\mathcal{C}^k$ (i.e., k times continuously differentiable) functions on $\mathbb{R}^n$, where $k \in \mathbb{N} \cup \{\infty\}$. Then X is ($\mathcal{F}$-) semicoherent if and only if $\mathcal{C}^\infty(X)$ is the intersection $\bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X)$ of all finite differentiability classes [Bierstone and Milman 1998; Bierstone et al. 1996]).

Question. Are restricted subpfaffian sets semicoherent? (A closed *restricted subpfaffian set* is a proper projection of a semianalytic set that is defined using *Pfaffian functions* in the sense of Khovanskii [1991]; compare [Wilkie 1996].)

3. Gabrielov's Theorem

Theorem 3.1. *Let $y = \varphi(x)$, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$, denote a real-analytic (or complex-analytic) mapping defined in a neighbourhood of a point a. Set $b = \varphi(a)$. Then the following conditions are equivalent*:

(1) $r_a(\varphi) = r_a^{\mathcal{F}}(\varphi)$; *i.e., there are "sufficiently many" formal relations.*

(2) $r_a(\varphi) = r_a^{\mathcal{F}}(\varphi) = r_a^{\mathcal{A}}(\varphi)$; *i.e., φ is regular at a.*

(3) *Composite function property*:

$$\mathcal{O}_a \cap \hat{\varphi}_a^*(\widehat{\mathcal{O}}_b) = \varphi_a^*(\mathcal{O}_b).$$

(4) *Linear equivalence of topologies. Let $R := \widehat{\mathcal{O}}_b / \operatorname{Ker} \hat{\varphi}_a^*$ and $R' := \widehat{\mathcal{O}}_a$, so there is a natural inclusion of local rings $R \hookrightarrow R'$. Let $\underline{m}$ and $\underline{m}'$ denote the maximal ideals of R and R', respectively. Then there exist $\alpha, \beta \in \mathbb{N}$ such that, for all k,*

$$(\underline{m}')^{\alpha k+\beta} \;\cap\; R \subset \underline{m}^k.$$

The composite function property (3) concerns the solution of an equation $f(x) = g\bigl(\varphi(x)\bigr)$, where f is a given analytic function at a and g is the unknown; (3) says that if there is a formal power series solution g, then there is also an analytic solution. Condition (4) concerns the *$\underline{m}$-adic* (or *Krull*) *topologies* of the local rings. (The powers $\underline{m}^k$ of the maximal ideal $\underline{m}$ of R form a fundamental system of neighbourhoods of 0 for the $\underline{m}$-adic topology.) Clearly, $\underline{m}^k \subset (\underline{m}')^k \cap R$ for all k, so (4) implies that the $\underline{m}$-adic topology of R coincides with its $\underline{m}'$-adic topology as a subspace of R'.

Gabrielov [1973] proved that (1) $\iff$ (2) and (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (2) follows from [Becker and Zame 1979] and [Milman 1978], and (2) $\iff$ (4) is due to Izumi [1986], Rees [1989] and Spivakovsky [1990].

Chevalley estimate. Condition (4) is related to the following elementary lemma of Chevalley [1943, § II, Lemma 7] (compare [Bierstone and Milman 1998, Lemma 5.2]).

LEMMA 3.2. (We use the notation of Theorem 3.1.) *For all* $k \in \mathbb{N}$, *there exists* $l \in \mathbb{N}$ *such that if* $G \in \widehat{\mathcal{O}}_b$ *and* $\hat{\varphi}_a^*(G) \in \underline{\hat{m}}_a^{l+1}$, *then* $G \in \operatorname{Ker} \hat{\varphi}_a^* + \underline{\hat{m}}_b^{k+1}$.

Given $k \in \mathbb{N}$, let $l_{\varphi^*}(a, k)$ denote the least l satisfying Chevalley's lemma. We call $l_{\varphi^*}(a, k)$ a *Chevalley estimate*. Condition (4) of Theorem 3.1 means there is a *linear Chevalley estimate* $l_{\varphi^*}(a, k) \le \alpha k + \beta'$ (where $\beta' = \alpha + \beta - 1$).

QUESTION. Suppose that $\varphi\colon M \to \mathbb{R}^n$ is a regular mapping (Definition 2.4). Is there a uniform linear Chevalley estimate $l_{\varphi^*}(a, k) \le \alpha_L k + \beta_L$, where $L \subset M$ is compact and $a \in L$? This question is equivalent to a uniform version of a product theorem of Izumi [1985] and D. Rees [1989]; see [Wang 1995], where the question also is answered positively in a special case.

4. Semicoherent Sets

In this section, we characterize the class of semicoherent subanalytic sets: We describe several metric, algebro-geometric and differential properties of subanalytic sets that might seem of quite different natures, but that turn out each to be equivalent to semicoherence. The ideas and results here come from [Bierstone and Milman 1998; Bierstone et al. 1996]. Theorem 4.4 below can be viewed as a parallel to Gabrielov's theorem 3.1, but is expressed in terms of properties of a closed subanalytic set X (i.e., the image $X = \varphi(M)$ of a proper real-analytic mapping $\varphi\colon M \to \mathbb{R}^n$) rather than in terms of properties of φ. The composite function property (3) of Theorem 3.1 is replaced by an analogous property concerning composite differentiable functions. According to Theorem 4.4, the $\mathcal{C}^\infty$ composite function property depends on the way that the formal local ideals $\mathcal{F}_b(X) = \bigcap_{a \in \varphi^{-1}(b)} \operatorname{Ker} \hat{\varphi}_a^*$ vary with respect to $b \in X$. The analytic composite function property (Theorem 3.1(3)), on the other hand, depends on the relationship between the convergent and formal ideals $\operatorname{Ker} \varphi_a^*$ and $\operatorname{Ker} \hat{\varphi}_a^*$. Theorem 4.4 shows that spaces of differentiable functions are natural function spaces on subanalytic sets.

$\mathcal{C}^\infty$ **composite function problem** (Thom, Glaeser). Let M denote a real-analytic manifold and $\varphi\colon M \to \mathbb{R}^n$ a proper real-analytic mapping. Suppose that $f\colon M \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function. Under what conditions is f a composite $f(x) = g\big(\varphi(x)\big)$, where g is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n$?

An obvious necessary condition is that f be constant on the fibres $\varphi^{-1}(b)$, where $b \in X := \varphi(M)$.

EXAMPLES 4.1 ($\mathcal{C}^\infty$ INVARIANTS OF A GROUP ACTION). In the early 1940's, Whitney proved that every $\mathcal{C}^\infty$ even function $f(x)$ (of one variable) can be written $f(x) = g(x^2)$, where g is $\mathcal{C}^\infty$ [Whitney 1943]. (See Example 4.2 below.) Whitney's result is the earliest version of the $\mathcal{C}^\infty$ composite function theorem. About twenty years later, Glaeser (answering a question posed by Thom in connection with the $\mathcal{C}^\infty$ preparation theorem) showed that a $\mathcal{C}^\infty$ function $f(x_1, \dots, x_n)$ which is invariant under permutation of the coordinates can be expressed $f(x) = g\big(\sigma_1(x), \dots, \sigma_n(x)\big)$, where g is $\mathcal{C}^\infty$ and the $\sigma_i(x)$ are the elementary symmetric polynomials [Glaeser 1963]. G. W. Schwarz [1975] extended these results to a $\mathcal{C}^\infty$ analogue of Hilbert's classical theorem on polynomial invariants: Hilbert's theorem says that, on a linear representation of a compact Lie group, the algebra of invariant polynomials is finitely generated; i.e., there are finitely many invariant polynomials $p_1(x), \dots, p_r(x)$ such that any invariant polynomial $f(x)$ can be written $f(x) = g\big(p_1(x), \dots, p_r(x)\big)$, where g is a polynomial. Schwarz's theorem asserts that a $\mathcal{C}^\infty$ invariant function $f(x)$ can be expressed in the same way, with g $\mathcal{C}^\infty$.

Formal composition. In each of Examples 4.1, f is constant on the fibres of the mapping (given by the basic invariant polynomials). In general, however, not every $\mathcal{C}^\infty$ function f that is constant on the fibres of a proper real-analytic mapping $\varphi\colon M \to \mathbb{R}^n$ can be expressed as a composite $f = g \circ \varphi$, where g is $\mathcal{C}^\infty$. (Consider, for example, $\varphi(x) = x^3$, $f(x) = x$.) There is a necessary formal condition [Glaeser 1963]: The Taylor expansions of f along any fibre $\varphi^{-1}(b)$ are the pull-backs of a formal power series centred at b; i.e., $f \in \big(\varphi^* C^\infty(\mathbb{R}^n)\big)\widehat{\ }$, where

$$\big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)\widehat{\ } := \left\{ \begin{matrix} f \in \mathcal{C}^\infty(M) : \text{for all } b \in \varphi(M), \text{ there exists } G_b \in \\ \widehat{\mathcal{O}}_b = \mathbb{R}[\![y-b]\!] \text{ such that } \hat{f}_a = \hat{\varphi}_a^*(G_b), \text{ for all } a \in \varphi^{-1}(b) \end{matrix} \right\}.$$

(Here $\hat{f}_a$ denotes the element of $\widehat{\mathcal{O}}_a$ induced by f: the formal Taylor expansion of f at a, with respect to any local coordinate system.) The functions in $\big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)\widehat{\ }$ are "formally composites with φ". (In each of Examples 4.1, the hypothesis implies formal composition.)

It is easy to see that $\big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)\widehat{\ }$ contains the closure of $\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)$ in $\mathcal{C}^\infty(M)$ (with respect to the $\mathcal{C}^\infty$ topology); in fact, $\big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)\widehat{\ }$ is closed [Bierstone et al. 1996, Corollary 1.4]. (For a definition of the $\mathcal{C}^\infty$ topology, see Question (2) following Theorem 4.4 below.) The *composite function property*

$$\varphi^*\mathcal{C}^\infty(\mathbb{R}^n) = \big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)\widehat{\ }$$

depends only on the image $X = \varphi(M)$ and holds, for example, if X is Nash-subanalytic [Bierstone and Milman 1982]. (The latter and [Bierstone and Milman 1987a; 1987b] are the sources of the ideas involved in Theorem 4.4.)

EXAMPLE 4.2 (PROOF OF WHITNEY'S THEOREM ON $\mathcal{C}^\infty$ EVEN FUNCTIONS). Suppose that $f(x)$ is a $\mathcal{C}^\infty$ function that is even; equivalently, f is formally a composite with $y = x^2$. We can assume that f is *flat at* 0 (i.e., f vanishes at 0 together with its derivatives of all orders) as follows: $\hat{f}_0(x) = H(x^2)$, where

$H \in \mathbb{R}[\![y]\!]$. By a classical lemma of E. Borel, there exists $h(y)$ of class $\mathcal{C}^\infty$ such that $\hat{h}_0 = H$. We can replace $f(x)$ by $f(x) - h(x^2)$.

Now let $g(y) = f(\sqrt{y})$, $y > 0$. Differentiating repeatedly, we have

$$\begin{pmatrix} f(x) \\ f'(x) \\ f''(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2x & 0 & \cdots \\ 0 & 2 & 4x^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} g(x^2) \\ g'(x^2) \\ g''(x^2) \\ \vdots \end{pmatrix}.$$

For any l, the determinant of the $l \times l$ upper left-hand block of the matrix is $c_l x^{l(l-1)/2}$, where $c_l > 0$. By Cramer's rule, since f is flat at 0, we have $\lim_{y\to 0+} g^{(k)}(y) = 0$, for all k. By L'Hôpital's rule (compare [Spivak 1994, Chapter 11, Theorem 7]), g extends to a $\mathcal{C}^\infty$ function that is flat at 0.

Approach to the composite function problem. Our point of view is similar to that in Example 4.2, and shows how the various properties of semicoherent sets enter into the composite function problem. Let $\varphi\colon M \to \mathbb{R}^n$ be a proper real-analytic mapping. Suppose $f \in \big(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\big)^\frown$. Then, for every $b \in X = \varphi(M)$, there exists $G_b \in \widehat{\mathcal{O}}_b$ such that $\hat{f}_a = G_b \circ \hat{\varphi}_a$, for all $a \in \varphi^{-1}(b)$. But in general G_b is uniquely determined only modulo $\mathcal{F}_b(X)$. A choice of a complementary subspace V_b to $\mathcal{F}_b(X)$ (i.e., $\widehat{\mathcal{O}}_b = \mathcal{F}_b(X) \oplus V_b$) provides a unique determination of G_b.

The equation $\hat{f}_a = G \circ \hat{\varphi}_a$, where $a \in \varphi^{-1}(b)$, implies that $\hat{f}_{a'} = G \circ \hat{\varphi}_{a'}$, for all a' in the same connected component of $\varphi^{-1}(b)$ as a [Bierstone and Milman 1998, Lemma 5.1]. Therefore, to find G_b as above, it is enough to solve the system of equations

$$\hat{f}_{a^i} = G_b \circ \hat{\varphi}_{a^i}, \quad \text{for } i = 1, \dots, s, \tag{4.3}$$

where there is at least one a^i in every component of $\varphi^{-1}(b)$. Since φ is a proper analytic mapping, there is a uniform bound on the number of connected components of a fibre $\varphi^{-1}(b)$, over any compact subset of the target.

We argue by successively flattening over a stratification $X = \cup X_j$ (as in Example 4.2). To guarantee that the G_b are Taylor expansions of a $\mathcal{C}^\infty$ function (at least along a stratum), we need to stratify so that the V_b can be chosen independent of $b \in X_j$, and invariant under formal differentiation in $\widehat{\mathcal{O}}_b = \mathbb{R}[\![y - b]\!]$. These properties hold on a stratification by the "diagram of initial exponents" $\mathcal{N}\big(\mathcal{F}_b(X)\big)$ (to be described below).

In general, it is not true that in (4.3) the coefficients of G_b of order $\leq k$ are determined (modulo $\mathcal{F}_b(X)$) by the coefficients of the $\hat{f}_{a^i}$ of order $\leq k$. (Even in Example 4.2 above, $l = 2k$ derivatives of f at 0 are needed to determine k derivatives of a formal solution H_0.) A uniform Chevalley estimate provides a uniform bound $l = l(k, K)$ on the number of formal derivatives of the $\hat{f}_{a^i}$, for $i = 1, \dots, s$, that are needed to determine the derivatives of order $\leq k$ of G_b mod $\mathcal{F}_b(X)$, for b in a compact subset K of $\mathbb{R}^n$.

We can then solve the composite function problem inductively over a stratification by the diagram, using Cramer's rule and a subanalytic version of L'Hôpital's rule or Hestenes's Lemma [Bierstone and Milman 1982, Corollary 8.2; Bierstone et al. 1996, Proposition 3.4], in a manner similar to Example 4.2.

Chevalley estimate. Let $\varphi\colon M \to \mathbb{R}^n$ be a proper real-analytic mapping and let $X = \varphi(M)$. Theorem 4.4 involves a variant of the Chevalley estimate (as defined in Section 3) for a fibre, or for the image X. For all $b \in X$ and $k \in \mathbb{N}$, we define

$$l_{\varphi^*}(b,k) := \min \left\{ \begin{array}{l} l \in \mathbb{N} : \text{if } G \in \widehat{\mathcal{O}}_b \text{ and } \hat{\varphi}_a^*(G) \in \underline{\hat{m}}_a^{l+1} \text{ for} \\ \text{all } a \in \varphi^{-1}(b), \text{ then } G \in \underline{\hat{m}}_b^{k+1} + \mathcal{F}_b(X) \end{array} \right\},$$

$$l_X(b,k) := \min \left\{ \begin{array}{c} l \in \mathbb{N} : \text{if } G \in \widehat{\mathcal{O}}_b \text{ and } |T_b^l G(y)| = o(|y-b|^l), \\ \text{where } y \in X, \text{ then } G \in \underline{\hat{m}}_b^{k+1} + \mathcal{F}_b(X) \end{array} \right\},$$

where $T_b^l G(y)$ denotes the Taylor polynomial of order l of G. Then $l_{\varphi^*}(b,k) < \infty$ because, if $\underline{a} = (a^1, \ldots, a^s)$, where each $a^i \in \varphi^{-1}(b)$ and some a^i belongs to each component of $\varphi^{-1}(b)$, then $l_{\varphi^*}(b,k) < l_{\varphi^*}(\underline{a},k)$, where

$$l_{\varphi^*}(\underline{a},k) := \min \left\{ \begin{array}{l} l \in \mathbb{N} : \text{if } G \in \widehat{\mathcal{O}}_b \text{ and } \hat{\varphi}_{a^i}^*(G) \in \underline{\hat{m}}_{a^i}^{l+1} \text{ for} \\ i = 1, \ldots, s, \text{ then } G \in \bigcap_i \operatorname{Ker} \hat{\varphi}_{a^i}^* + \underline{\hat{m}}_b^{l+1} \end{array} \right\},$$

and $l_{\varphi^*}(\underline{a},k) < \infty$ as in Lemma 3.2. On the other hand, l_{φ^*} and l_X are equivalent on compact subsets of X in the sense that, for every compact $K \subset X$, there exists r_K ($r_K \geq 1$) such that

$$l_X(b,\,\cdot\,) \;\leq\; l_{\varphi^*}(b,\,\cdot\,) \;\leq\; r_K l_X(b,\,\cdot\,),$$

$b \in K$. These inequalities are consequences of the two metric inequalities

$$\begin{aligned} |\varphi(x) - b| &\leq c_\varphi(K) d(x,a), && \text{for } b \in K, \\ d\big(x, \varphi^{-1}(b)\big)^r &\leq c_\varphi(b,K) |\varphi(x) - b|, && \text{for } b \in K, \end{aligned}$$

where $r \geq 1$ and $d(\,\cdot\,,\,\cdot\,)$ denotes a locally Euclidean metric on M [Bierstone and Milman 1998, Lemma 6.5]. The first of these metric inequalities is simple; the second is an important estimate of Tougeron [1971].

Diagram of initial exponents. Let I be an ideal in the ring of formal power series $\mathbb{R}[\![y-b]\!] = \mathbb{R}[\![y_1 - b_1, \ldots, y_n - b_n]\!]$. The *diagram of initial exponents* $\mathcal{N}(I) \subset \mathbb{N}^n$ is a combinatorial representation of I, in the spirit of the classical Newton diagram of a formal power series;

$$\mathcal{N}(I) := \big\{ \exp G : \; G \in I \setminus \{0\} \big\},$$

where $\exp G$ denotes the smallest exponent β of a monomial

$$(y-b)^\beta = (y_1 - b_1)^{\beta_1} \cdots (y_n - b_n)^{\beta_n}$$

with nonzero coefficient in the expansion of G ("smallest" with respect to the lexicographic order of $(|\beta|, \beta_1, \dots, \beta_n)$, where $|\beta| = \beta_1 + \cdots + \beta_n$). The diagram $\mathcal{N} = \mathcal{N}(I)$ has the form $\mathcal{N} = \mathcal{N} + \mathbb{N}^n$ (since I is an ideal); therefore, there is a smallest finite subset $\mathcal{V}$ of $\mathbb{N}^n$ such that $\mathcal{N} = \mathcal{V} + \mathbb{N}^n$. We call the elements of $\mathcal{V}$ the *vertices* α^j of $\mathcal{N}$.

Set $\operatorname{supp} G := \{\beta : G_\beta \neq 0\}$, where $G = \sum G_\beta (y-b)^\beta$. Hironaka's *formal division algorithm* [Hironaka 1964; Bierstone and Milman 1987a, Theorem 6.2; 1998, Theorem 3.1] shows that

$$\mathbb{R}[\![y-b]\!] = I \oplus \mathbb{R}[\![y-b]\!]^{\mathcal{N}(I)},$$

where

$$\mathbb{R}[\![y-b]\!]^{\mathcal{N}(I)} := \bigl\{G : \operatorname{supp} G \cap \mathcal{N}(I) = \varnothing\bigr\},$$

and that, if we write

$$(y-b)^{\alpha_j} = F^j(y) + R^j(y),$$

where $F^j(y) \in I$ and $R^j(y) \in \mathbb{R}[\![y-b]\!]^{\mathcal{N}(I)}$, for every vertex α^j, then $\{F^j\}$ is a set of generators of I [Bierstone and Milman 1987a, Corollary 6.8; 1998, Corollary 3.2]. We call $\{F^j\}$ the *standard basis* of I. (It is uniquely determined by the condition that $F^j - (y-b)^{\alpha^j} \in \mathbb{R}[\![y-b]\!]^{\mathcal{N}(I)}$, for each j.) Since $\mathcal{N}(I) + \mathbb{N}^n = \mathcal{N}(I)$, it follows that $\mathbb{R}[\![y-b]\!]^{\mathcal{N}(I)}$ is stable with respect to formal differentiation.

The diagram of initial exponents determines many important algebraic invariants of the ring $\mathbb{R}[\![y-b]\!]/I$; for example, the *Hilbert–Samuel function*, defined for $k \in \mathbb{N}$ by

$$H_I(k) := \dim_{\mathbb{R}} \frac{\mathbb{R}[\![y-b]\!]}{I + (y-b)^{k+1}} = \#\{\beta \in \mathbb{N}^n \setminus \mathcal{N}(I) : \ |\beta| \leq k\},$$

where $(y-b)$ here denotes the maximal ideal of $\mathbb{R}[\![y-b]\!]$.

Characterization of semicoherent sets. Suppose that $Z \subset X$ are closed analytic subsets of $\mathbb{R}^n$. If $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{C}^k(X;Z)$ denotes the algebra of restrictions to X of $\mathcal{C}^k$ functions on $\mathbb{R}^n$ that are *k-flat* on Z (i.e., that vanish on Z together with all partial derivatives of orders at most k). If φ: $M \to \mathbb{R}^n$ is a proper real-analytic mapping such that $\varphi(M) = X$, we set

$$\bigl(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n;Z)\bigr)\widehat{\ } := \bigl(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n)\bigr)\widehat{\ } \cap \mathcal{C}^\infty\bigl(M;\varphi^{-1}(Z)\bigr).$$

THEOREM 4.4 [Bierstone and Milman 1998; Bierstone et al. 1996]. *The following conditions are equivalent*:

(1) *X is semicoherent rel Z.*

(2) *Composite function property. If φ: $M \to \mathbb{R}^n$ is a proper real-analytic mapping such that $X = \varphi(M)$, then*

$$\varphi^*\mathcal{C}^\infty(\mathbb{R}^n;Z) = \bigl(\varphi^*\mathcal{C}^\infty(\mathbb{R}^n;Z)\bigr)\widehat{\ }.$$

(3) $\mathcal{C}^\infty(X;Z) = \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(X,Z)$.

(4) *Uniform Chevalley estimate. For every compact subset K of X, there is a function $l_K\colon \mathbb{N} \to \mathbb{N}$ such that*

$$l_X(b,k) \le l_K(k), \quad \text{for all } b \in K \cap (X \setminus Z).$$

(5) *Stratification by the diagram of initial exponents. X has a (locally finite) subanalytic stratification $X = \bigcup X_i$ such that Z is a union of strata and $\mathcal{N}_b := \mathcal{N}\big(\mathcal{F}_b(X)\big)$ is constant on every stratum outside Z.*

(6) *Stratification by the Hilbert–Samuel function.*

The conditions of Theorem 4.4 are satisfied, for example, if X is a closed subanalytic set and Z is the set of non-Nash points of X.

Conditions (5) and (6) of the theorem can be replaced by conditions of *subanalytic semicontinuity* that are *a priori* stronger. Subanalytic semicontinuity of $\mathcal{N}_b$, for example, means adding to (5) the condition that, if $X_j \subset \overline{X}_i$, $X_j \not\subset Z$, then $\mathcal{N}_j \ge \mathcal{N}_i$, where, for each i, $\mathcal{N}_i$ denotes the value of $\mathcal{N}_b$ on X_i, and $\ge$ is a natural ordering on the set of all possible diagrams. See [Bierstone and Milman 1998].

If $X = \cup X_i$ is a stratification by the diagram, as in (5), then X satisfies the formal semicoherence property along every stratum X_i outside Z; in fact, the standard basis

$$F^{ij}(b,y) = (y-b)^{\alpha^{ij}} + \sum_{\beta \in \mathbb{N}^n \setminus \mathcal{N}_i} f_{ij,b}(b)(y-b)^{\beta}$$

(where $\{\alpha^{ij}\}$ denotes the vertices of $\mathcal{N}_i$) provides a semicoherent structure [Bierstone and Milman 1998, § 9].

Question 4.5. Suppose that X is a Nash-subanalytic set. Is there a uniform linear Chevalley estimate $l_X(b,k) \le \alpha_K k + \beta_K$, where $K \subset X$ is compact and $b \in K$? (A variant of the question in Section 3 above.)

Question 4.6. Functional-analytic characterization of "tame"; for example, characterization of semicoherent sets by the extension property. Let X be a closed subanalytic subset of $\mathbb{R}^n$. We say that X has the *extension property* if the restriction mapping $\mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(X)$ has a continuous linear splitting (or right inverse) E. If X is semicoherent, then there is an extension operator E [Bierstone and Milman 1998, Theorem 1.23].

The topology of $\mathcal{C}^\infty(\mathbb{R}^n)$ is defined by a system of seminorms

$$\|f\|_k^K := \sup_{\substack{y \in K \\ |\beta| \le k}} \left| \frac{\partial^{|\beta|} f(y)}{\partial y^\beta} \right|,$$

where $k \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ is compact. The topology of $\mathcal{C}^\infty(X)$ is defined by the induced quotient seminorms $\|g\|_l^L := \inf\{\|f\|_l^L\colon f \in \mathcal{C}^\infty(\mathbb{R}^n),\ f|X = g\}$. If X is semicoherent and $E\colon \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is an extension operator, then for every $k \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ compact, there exist $l = l(K,k) \in \mathbb{N}$, $L = L(K,k)$

compact, and a constant $c = c(K,k)$ such that $\|E(g)\|_k^K \le c\|g\|_l^L$, for all $g \in \mathcal{C}^\infty(X)$. We do not have a precise estimate on $l = l(K,k)$, in general. But, for example, if $X = \overline{\operatorname{int} X}$, then there is an extension operator with a linear estimate $l(K,k) = \lambda k$, where $\lambda = \lambda(K)$ [Bierstone 1978]. In the direction converse to "semicoherence implies the extension property", we can prove that if X has an extension operator with an estimate $l(0,K) = 0$ on the zeroth seminorms for every compact K, then there is a uniform Chevalley estimate $l_X(b,k) \le l(K,2k)$, $b \in K$ [Bierstone and Milman 1998, Proposition 1.24]. In view of Theorem 4.4, it is therefore interesting to ask: Does semicoherence imply the extension property with $l(\,\cdot\,,0) = 0$?

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